Specializations of Jordan superalgebras

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Abstract. We construct universal associative enveloping algebras for a large class of Jordan superalgebras.

1. Introduction

Let \( F \) be a ground field of characteristic \( \neq 2 \). A (linear) Jordan algebra is a vector space \( J \) with a binary bilinear operation \( (x, y) \mapsto xy \) satisfying the following identities:

\begin{align*}
(J1) & \quad xy = yx \\
(J2) & \quad (x^2y)x = x^3(yx)
\end{align*}

For an element \( x \in J \) let \( R(x) \) denote the right multiplication \( R(x) : a \mapsto ax \) in \( J \). If \( x, y, z \in J \) then by \( \{x, y, z\} \) we denote their Jordan triple product \( \{xy, z\} = (xy)z + x(yz) - y(xz) \).

A Jordan algebra \( J \) is called special if it is embeddable into an algebra of type \( A^{(\pm)} \), where \( A \) is an associative algebra. The algebra \( H_3(\mathbb{O}) \) is exceptional. A homomorphism \( J \to A^{(\pm)} \) is called a specialization of a Jordan algebra \( J \). N. Jacobson [3] introduced the notion of a universal associative enveloping algebra \( U = U(J) \) of a Jordan algebra \( J \) and showed that the category of specializations of \( J \) is equivalent to the category of homomorphisms of the associative algebra \( U(J) \).

Let \( V \) be a Jordan bimodule over the algebra \( J \) (see [3]). We call \( V \) a one-sided bimodule if \( \{J, V, J\} = (0) \). In this case, the mapping \( \alpha \mapsto 2R_{\alpha} \in \text{End}_F V \) is a specialization. The category of one-sided bimodules over \( J \) is equivalent to the category of right (left) \( U(J) \)-modules.


In this paper we study specializations and one-sided bimodules of Jordan superalgebras. Let us introduce the definitions.

By a superalgebra we mean a \( \mathbb{Z}/2\mathbb{Z} \)-graded algebra \( A = A_\sigma + A_\tau \). We define \( [\alpha] = 0 \) if \( \alpha \in A_\sigma \) and \( [\alpha] = 1 \) if \( \alpha \in A_\tau \).
For instance, if $V$ is a vector space of countable dimension, and $G(V) = G(V)_\mathbb{R} + G(V)_\mathbb{I}$ is the Grassmann algebra over $V$, that is, the quotient of the tensor algebra over the ideal generated by the symmetric tensors, then $G(V)$ is a superalgebra. Its even part is the linear span of all products of even length and the odd part is the linear span of all products of odd length.

If $A$ is a superalgebra, its Grassmann enveloping algebra is the subalgebra of $A \otimes G(V)$ given by $G(A) = A_\mathbb{R} \otimes G(V)_\mathbb{R} + A_\mathbb{I} \otimes G(V)_\mathbb{I}$.

Let $\mathcal{V}$ be a homogeneous variety of algebras, that is, a class of $F$-algebras satisfying a certain set of homogeneous identities and all their partial linearizations (see [20]).

**Definition 1** A superalgebra $A = A_\mathbb{R} + A_\mathbb{I}$ is called a $\mathcal{V}$ superalgebra if $G(A) \in \mathcal{V}$.

C. T. C. Wall [19] showed that every simple finite-dimensional associative superalgebra over an algebraically closed field $F$ is isomorphic to the superalgebra

$$M_{m,n}(F) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, A \in M_m(F), D \in M_n(F) \right\} + \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, B \in M_{m\times n}(F), C \in M_{n\times m}(F) \right\}$$

or to the superalgebra

$$P(n) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, A \in M_n(F) \right\} + \left\{ \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix}, B \in M_n(F) \right\}.$$
6) The 3-dimensional Kaplansky superalgebra, \( K_3 = Fe + (Fx + Fy) \), with the multiplication \( e^2 = e, \quad ex = \frac{1}{2}x, \quad ey = \frac{1}{2}y, \quad [x,y] = e \).

7) The 1-parametric family of 4-dimensional superalgebras \( D_t \) is defined as \( D_t = (Fe_1 + Fe_2) + (Fx + Fy) \) with the product: \( e_1^2 = e_1e_1 = 0, \quad e_1xe_1 = \frac{1}{2}x, \quad ey = e_1 + te_2, \quad i = 1, 2 \).

The superalgebra \( D_t \) is simple if \( t \neq 0 \). In the case \( t = -1 \), the superalgebra \( D_{-1} \) is isomorphic to \( M_{1,1}(F) \).

8) The 10-dimensional Kac superalgebra (see [5]) has been proved to be exceptional in [15]. In characteristic 3 this superalgebra is not simple, but it has a subalgebra of dimension 9 that is simple (the degenerated Kac superalgebra. There are two other examples of simple Jordan superalgebras in characteristic 3, both of them exceptional (see [16]).

9) We will consider now Jordan superalgebras defined by a bracket. If \( A = A_0 + A_1 \) is an associative commutative superalgebra with a bracket on \( A \), the Kantor Double of \( A \) is \( A \otimes Z[\partial] \) with the multiplication in \( A \) given by:

\[
\partial a \otimes b = [a,b], \quad \partial (ab) = \partial a \otimes b + (-1)^{|a||b|}a \otimes \partial b,
\]

where \( [a,b] = ab - ba \). The Kantor Double of the Grassmann algebra with the bracket \( [\cdot, \cdot] \) is a Jordan superalgebra. Every Poisson bracket is a Jordan bracket.

10) Let \( Z \) be a unital associative commutative algebra with a derivation \( \partial : Z \to Z \). Consider the superalgebra \( CK(Z, \partial) = A + M \), where \( A = J_0 = Z + \sum_{i=1}^{3} w_i Z, \quad M = J_1 = xZ + \sum_{i=1}^{3} x_i Z \) are free \( Z \)-modules of rank 4. The multiplication on \( A \) is \( Z \)-linear and \( w_iw_j = 0, i \neq j, \quad w_1^2 = w_2^2 = 1, \quad w_3^2 = -1 \).

Denote \( x_{i\otimes i} = 0, \quad x_{1\otimes 2} = -x_{2\otimes 1} = x_3 \quad x_{1\otimes 3} = -x_{3\otimes 1} = x_2 \quad -x_{2\otimes 3} = x_{3\otimes 2} = x_1 \).

The bimodule structure and the bracket on \( M \) are defined via the following tables:

<table>
<thead>
<tr>
<th>( g )</th>
<th>( w_j g )</th>
<th>( xg )</th>
<th>( x_j g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( xf )</td>
<td>( x(fg) )</td>
<td>( x_j(fg^d) )</td>
<td>( xf )</td>
</tr>
<tr>
<td>( xi f )</td>
<td>( xi(fg) )</td>
<td>( xi_j(fg) )</td>
<td>( xi f )</td>
</tr>
</tbody>
</table>

The superalgebra \( CK(Z, \partial) \) is simple if and only if \( Z \) does not contain proper \( \partial \)-invariant ideals.

In [5], [8] it was shown that simple finite dimensional Jordan superalgebras over an algebraically closed field \( F \) of zero characteristic are those of examples 1) - 8) and the Kantor Double (example 9) of the Grassmann algebra with the bracket \( \{ f, g \} = \sum (-1)^{|f||g|} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \xi_i} \).

The examples 9), 10) are related to infinite dimensional superconformal Lie superalgebras (see [6], [7]). In particular, the superalgebras \( CK(Z, \partial) \) correspond to an important superconformal algebra discovered in [1] and [2].

In [13] it was shown that the only simple finite dimensional Jordan superalgebras over an algebraically closed field of characteristic \( p > 2 \) with nonsemisimple even part are superalgebras (9), (10) built on truncated polynomials.
2. Universal enveloping algebras

In what follows the ground field \( F \) is assumed to be algebraically closed.

1. Let \( U' \) be a universal associative enveloping algebra of a special Jordan superalgebra \( J, u : J \to U' \) a universal specialization. The algebra \( U' \) is equipped with a natural superinvolution \( * \) leaving all elements from \( u(J) \) fixed. Then \( u(J) \subseteq H(U, *) \). We call a superalgebra \( J \) reflexive if \( u(J) = H(U, *) \).

**Theorem 1** All superalgebras of examples 1) - 4) are reflexive except the following ones: \( \text{M}_{i,1}^{(\pm)}(F) \), \( \text{Osp}(1,2) \cong D(-2), Q(2) \). Hence,

\[
\begin{align*}
U(M_{i,1}^{(\pm)}(F)) &\cong M_{m,n}(F) \oplus M_{m,n}(F) \quad \text{for } (m,n) \neq (1,1); \\
U(P^{(\pm)}(n)) &\cong P(n) \oplus P(n), \quad n \geq 2; \\
U(\text{Osp}(m,n)) &\cong M_{m,n}(F), \quad (m,n) \neq (1,2); \\
U(Q(n)) &\cong M_{m,n}(F), \quad n \geq 3.
\end{align*}
\]

2. Let \( Z \) be an associative commutative algebra with a derivation \( D : Z \to Z \). Let \( W = Z, D > 0 \) and let \( u : CK(Z, D) \to M_{2,2}(W) \) be the embedding found in [12].

The embedding \( u \) extends the embedding of Kantor doubles of brackets of vector type found in [14].

**Theorem 2** \( U(CK(Z, D)) = M_{2,2}(W) \). The embedding \( u \) is universal.

3. The superalgebra of \( CK(Z, D) \) spanned over \( F \) by the elements \( 1, w_1, w_2, w_3, x, x_1, x_2, x_3 \) is isomorphic to \( Q(2) \).

**Theorem 3** The restriction of the embedding \( u \) (see above) to \( Q(2) \) is a universal specialization;

\[
U(Q(2)) \cong M_{2,2}(F[t]),
\]

where \( F[t] \) is a polynomial algebra in one variable.

4. Let us describe the universal associative enveloping superalgebra of \( M_{1,1}(F) \). Consider the ring of polynomials and the field of rational functions in two variables, \( F[z_1, z_2] \subseteq F(z_1, z_2) \). Let \( K \) be the quadratic extension of \( F(z_1, z_2) \) generated by a root of the equation \( a^2 + a - z_1 z_2 = 0 \). Consider the subring \( A = F[z_1, z_2] + F[z_1, z_2]a \) and the subspaces \( M_{12} = F[z_1, z_2] + F[z_1, z_2]a^{-1} z_2, M_{21} = F[z_1, z_2]z_1 + F[z_1, z_2]a \) of \( K \). Then \( U = \left( \begin{array}{cc} A & M_{12} \\ M_{21} & A \end{array} \right) \) is a subring of \( M_{2}(K) \).

**Theorem 4** \( U(M_{1,1}(F)) \cong \left( \begin{array}{cc} A & M_{12} \\ M_{21} & A \end{array} \right) \). The mapping

\[
u : \left( \begin{array}{cc} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{array} \right) \to \left( \begin{array}{cc} \alpha_{11} & \alpha_{12} + \alpha_{21} a^{-1} z_2 \\ \alpha_{12} z_1 + \alpha_{21} a \end{array} \right)
\]

is a universal specialization.

5. Let \( V = V^0 + V^1 \) be a \( Z/2Z \)-graded vector space, \( \dim V^0 = m, \dim V^1 = 2m \); let \( <,> : V \times V \to F \) be a supersymmetric bilinear form on \( V \). The universal associative enveloping algebra of the Jordan algebra \( F1 + V_T \) is the Clifford algebra \( C1(m) = \{1, e_1, \ldots, e_m | e_i e_j + e_j e_i = 0, i \neq j, e_i^2 = 1 \} \) (see [3]). Assuming the generators \( e_1, \ldots, e_m \) to be odd, we get a \( Z/2Z \)-gradation on \( C1(m) \).

In \( V_T \) we can find a basis \( v_1, v_2, \ldots, v_n, w_1, w_2, w_3, w_4 \) such that \( < v_i, w_j > = \delta_{ij}, \quad < v_i, v_j > = < w_i, w_j > = 0 \). Consider the Weyl algebra \( W_n = (1, x_i, y_j, 1 \leq i \leq n) \) such that \( [x_i, y_j] = \delta_{ij}, [x_i, x_j] = [y_i, y_j] = 0 > \). Assuming
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$x_i, y_i, 1 \leq i \leq n$ to be odd, we make $W_n$ a superalgebra. The universal associative enveloping algebra of $F^1 + V$ is isomorphic to the (super)tensor product $Cl(n) \otimes_F W_n$.

6. Let $osp(1,2)$ denote the Lie subsuperalgebra of $M_{1,2}(F)$ which consists of skewsymmetric elements with respect to the orthosympletic superinvolution. Let $x,y$ be the standard basis of the odd part of $osp(1,2)$.

**Theorem 5** (I. Shestakov) The universal enveloping algebra of $K_3$ is isomorphic to $U(osp(1,2)/id([x,y]^2 - [x,y]))$, where $U(osp(1,2))$ is the universal associative enveloping algebra of $osp(1,2)$ and $id([x,y]^2 - [x,y])$ is the ideal of $U(osp(1,2))$ generated by $[x,y]^2 - [x,y]$.

Clearly, if $ch F = 0$ then $K_3$ does not have nonzero specializations that are finite dimensional algebras. If $ch F = p > 0$ then $K_3$ has such specializations.

7. Let us consider the superalgebras $D(t)$. We will assume that $t \neq -1,0,1$, because $D(-1) \simeq M_{1,1}(F)$; $D(0) \simeq K_3 + F$; $D(1)$ is a Jordan superalgebra of a superform.

**Theorem 6** (I. Shestakov) The universal enveloping algebra of $D(t)$ is isomorphic to $U(osp(1,2)/id([x,y]^2 - (1+t)[x,y] + t)$.

**Corollary 1** If $ch F = 0$ then all finite dimensional one-sided bimodules over $D(t)$ are completely reducible.

Indeed, it is known (see [4]) that finite dimensional representations of the Lie superalgebra $osp(1,2)$ are completely reducible.

Now we will assume that $ch F = 0$ and will classify irreducible finite dimensional one-sided bimodules over $D(t)$. Let us first consider four infinite dimensional Verma type right modules over $U(D(t))$. Each of these bimodules is generated by an even highest weight element $v$.

$V_1(t) = vU(d(t))$. Defining relations: $v(xy + yx) = (2t + 1)v, vy = 0, ve_1 = v, ve_2 = 0$. Basis: $v, vy, vx^i, i \geq 1$.

$V_2(t) = vU(d(t))$. Defining relations: $v(xy + yx) = (2t + 1)v, vy = 0, ve_1 = v, ve_2 = 0$. Basis: $v, vx^i, i \geq 1$.

Changing parity we get two new bimodules $V_1(t)^{op}$ and $V_2(t)^{op}$.

Each of these bimodules has the unique irreducible homomorphism image $W_1(t)$ or $W_2(t)$ or $W_1(t)^{op}$ or $W_2(t)^{op}$ respectively.

**Theorem 7** If $t = \frac{-m+1}{m}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $W_1(t)$ and $W_1(t)^{op}$.

If $t = \frac{m}{m+1}$, $m \geq 1$, then $D(t)$ has two irreducible finite dimensional one sided bimodules $W_2(t)$ and $W_2(t)^{op}$.

If $t$ can not be represented as $\frac{-m+1}{m}$ or $\frac{m}{m+1}$, where $m$ is a positive integer, then $D(t)$ does not have nonzero finite dimensional specializations.

If $ch F = p > 2$ then for an arbitrary $t$ the superalgebra $D(t)$ can be embedded into a finite dimensional associative superalgebra. It suffices to notice that $D(t) \subseteq CK(F[t]^{op}, d/dt)$.

5
References


