Complemented copies of $c_0$ in $C_0 (\Omega)$

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Abstract. In this note we consider a class of locally compact Hausdorff topological spaces $\Omega$ with the property that the Banach space $C_0 (\Omega)$ of all scalarly valued continuous functions defined on $\Omega$ vanishing at infinity, equipped with the supremum-norm, contains a norm-one complemented copy of $c_0$ whereas $C (\beta \Omega)$ contains a linearly isometric copy of $\ell_\infty$.

Copias complementadas de $c_0$ en $C_0 (\Omega)$

Resumen. En esta nota consideramos una clase de espacios topológicos de Hausdorff localmente compactos $\Omega$ con la propiedad de que el espacio de Banach $C_0 (\Omega)$ de todas las funciones continuas con valores escalares definidas en $\Omega$ que se anulan en el infinito, equipado con la norma suprema, contiene una copia de $c_0$ norma-uno complementada, mientras que $C (\beta \Omega)$ contiene una copia de $\ell_\infty$ linealmente isométrica.

1. Preliminaries

Some known results are given in this preliminaires in order to make the reading easier and also to present some new viewpoints. All topological spaces considered in this paper will be Hausdorff. In what follows $K$ will stand for a compact topological space, $\Omega$ for a locally compact topological space and $X$ for a Banach space. The Banach space over the field $\mathbb{K}$ of the real or complex numbers of all $\mathbb{K}$-valued continuous functions defined on $\Omega$ equipped with the supremum norm $\|f\|_\infty = \sup \{ |f(\omega)| : \omega \in K \}$ will be represented by $C(K, \mathbb{K})$ [by $C(K)$ if $X = \mathbb{K}$]. The Banach space over $\mathbb{K}$ of all continuous functions $f : \Omega \to \mathbb{K}$ vanishing at infinity (that is, for each $\epsilon > 0$ there is a compact set $K_{\epsilon, f} \subseteq \Omega$ such that $|f(\omega)| < \epsilon$ for each $\omega \in \Omega - K_{\epsilon, f}$) will be denoted by $C_0 (\Omega)$. The linear subspace of $C_0 (\Omega)$ consisting of all those functions $f$ of compact support, $\text{supp} \ f$, will be represented by $C_c (\Omega)$. A compact space $K$ is called Eberlein compact if $K$ is homeomorphic to a weakly compact set of a Banach space. If $I$ is an index set and $\sum (I)$ denotes the subset of $[0, 1]^I$ of all those functions of countable support, a compact space $K$ is called Valdivia compact [3, 7] if there exists a set $I$ such that $K$ is homeomorphic to a subset $K_0$ of $[0, 1]^I$ with the property that $K_0 \cap \sum (I)$ is dense in $K_0$. Every compact metric space is Eberlein compact, and every Eberlein compact is Valdivia compact [3, Theorem 7.2]. A Banach space $X$ is said to be a Grothendieck space if each bounded linear operator $T : X \to c_0$ is weakly compact or, equivalently [4, Chapter VII. Exercise 4], if each weak* null sequence in $X^*$ is weakly null. Concerning complemented copies of $c_0$ in $C(K, X)$, we have the following [1, 6].

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Theorem 1 (Cembranos-Freniche) If \( K \) is infinite and \( X \) infinite-dimensional, then \( C(K, X) \) contains a complemented subspace isomorphic to \( c_0 \).

Relative to copies of \( \ell_\infty \) one has Drewnowski’s fundamental result [5].

Theorem 2 (Drewnowski) \( C(K, X) \) contains a copy of \( \ell_\infty \) if and only if \( C(K) \) contains a copy of \( \ell_\infty \) or \( X \) contains a copy of \( \ell_\infty \).

When \( X \) is finite-dimensional the situation in \( C(K, X) \) becomes more troublesome. For instance, if \( \mathbb{N} \) denotes the set of the positive integers provided with the discrete topology and \( \beta \mathbb{N} \) stands for the Stone-Čech compactification of \( \mathbb{N} \), by the preceding theorems \( C(\beta \mathbb{N} \times \beta \mathbb{N}) \cong C(\beta \mathbb{N}, \ell_\infty) \) contains a complemented copy of \( c_0 \) and a copy of \( \ell_\infty \). There are no known both necessary and sufficient conditions for \( C(K) \) to contain a complemented copy of \( c_0 \) or a copy of \( \ell_\infty \). Nevertheless, the following results [2, 3, 4] are well known.

Theorem 3 (Grothendieck) If \( K \) is extremally disconnected, then \( C(K) \) contains no complemented copies of \( c_0 \).

Theorem 4 (Rosenthal) If \( K \) is extremally disconnected, then \( C(K) \) contains a copy of \( \ell_\infty \).

According to Borsuk-Dugundji’s extension theorem [11], which states that if \( M \) is a closed metrizable subspace of a compact space \( K \) there is a norm-one extension linear operator \( T \) from \( C(M) \) into \( C(K) \), then \( C(K) \) contains a complemented copy of \( C(M) \). In fact, the mapping \( P : C(K) \to T(C(M)) \) defined by \( Ph = T(h|_M) \) is a norm-one bounded linear projection operator from \( C(K) \) onto \( T(C(M)) \). In particular, if \( K \) contains a nontrivial convergent sequence \( \{t_n\} \) and \( t = \lim t_n \), setting \( M = \{t, t_n : n \in \mathbb{N}\} \) then \( M \) is a closed metrizable subspace of \( K \) and hence \( C(M) \) has a complemented copy in \( C(K) \). Since \( C(M) \) is isometric to \( c_0 \), the Banach space of all convergent sequences endowed with the supremum-norm, it follows that \( C(K) \) contains a complemented copy of \( c_0 \). This can be obtained straightforwardly without appealing to the Borsuk-Dugundji theorem as follows.

Proposition 1 Assume \( K \) is a compact topological space. If \( K \) contains a nontrivial convergent sequence, then \( C(K) \) contains a complemented copy of \( c_0 \).

Proof. Suppose that \( \{t_n\} \) is an injective convergent sequence in \( K \), and let \( t \) be its limit. Choose a sequence \( \{U_n\} \) of pairwise disjoint open subsets of \( K - \{t\} \) such that \( t_n \in U_n \) for each \( n \in \mathbb{N} \) and \( t_m \notin U_n \) if \( n \neq m \), and use Urysohn’s lemma to select \( f_n \in C(K) \), with \( 0 \leq f_n \leq 1 \), such that \( f_n(t_n) = 1 \) and \( \text{supp } f_n \subseteq U_n \). Then \( \{f_n\} \) is a normalized basic sequence in \( C(K) \) equivalent to the unit vector basis \( \{e_n\} \) of \( c_0 \) with 1 as basis constant. If \( \delta_\omega \) denotes the Dirac measure at \( \omega \in K \), since \( t_n \to t \) in \( K \) then \( \delta_{t_n} \to \delta_t \) weakly* in \( C(K) \), so that \( \{\delta_{t_n} - \delta_t\} \) is a weak* null sequence in \( C(K)^* \) such that \( \langle \delta_{t_n} - \delta_t, f_j \rangle = \delta_{t_n} f_j \) for each \( i, j \in \mathbb{N} \). So \( \{f_n\} \) is a complemented copy of \( c_0 \) as a consequence of [2, Proposition 1.1.2]. Explicitly, for each \( n \in \mathbb{N} \) define the linear functional \( u_n : C(K) \to \mathbb{K} \) by \( u_n(f) = \langle \delta_{t_n} - \delta_t, f \rangle \), and, given \( f \in C(K) \), note that \( |u_n(f)| \leq 2 \|f\|_\infty \). Consequently, the linear operator \( P : C(K) \to [f_n] \) defined by \( Pf = \sum_{n=1}^\infty u_n(f) f_n \) satisfies that \( \|Pf\|_\infty \leq \sup_{n \in \mathbb{N}} \|\sum_{i=1}^n u_i(f) f_i\| = \sup_{n \in \mathbb{N}} |u_n(f)| \leq 2 \|f\|_\infty \). Since \( \delta_t \in [f_n]^{-1} \), we have \( P\delta_t = \sum_{n=1}^\infty u_n(f_n) f_n = \sum_{n=1}^\infty (\delta_{t_n} - \delta_t, f_n) f_n = f_j \) for each \( j \in \mathbb{N} \), and thus \( P \) is a bounded linear projection operator from \( C(K) \) onto \( [f_n] \).

Corollary 1 If \( K \) is a compact space containing a nontrivial convergent sequence, then \( C(K) \) is not a Grothendieck space.

Proof. This is an obvious consequence of [1, Corollary 2].
Remark 1 If $K$ is an infinite Valdivia compact set in $[0, 1]^I$ and \( \{x_n\} \) is an injective sequence in $K \cap \sum (I)$, since $J := \bigcup_{n=1}^{\infty} \text{supp } x_n$ is countable there is a subsequence $\{x_{n_i}\}$ which converges to some $x \in K \cap \sum (I)$. So each infinite Valdivia compact set contains a nontrivial convergent sequence. On the other hand, Helly’s space $H$, i.e. the set of all nondecreasing functions of the product space $[0, 1]^I$, with $I = [0, 1]$, is an infinite compact, sequentially compact (hence $H$ contains a nontrivial convergent sequence), separable and nonmetrizable space [8, Chapter 5, Problem M], which is not Valdivia compact. Indeed, the set $\bigcup_{n=1}^{\infty} \text{supp } x_n$ is not dense in $H$.

In an infinite locally compact topological space $\Omega$ there are always two disjoint nonempty open sets such that one of them is infinite. Indeed, choose a nondense open subset $A$ of $\Omega$. If $\Omega - \overline{A}$ is infinite, take $A$ and $\Omega - \overline{A}$, otherwise take $\Omega - \overline{A}$ and $\overline{A}$. Consequently, in each infinite compact topological space $K$ there exists a sequence $\{U_n\}$ of nonempty pairwise disjoint open sets. So, selecting $t_n \in U_n$ and $f_n \in C(K)$ for each $n \in \mathbb{N}$ as in the proof of Proposition 1, then $\{f_n\}$ is a basic sequence in $C(K)$ equivalent to the unit vector basis of $c_0$. If $K$ is a Valdivia compact set and $Z$ is a separable subspace of $C(K)$, it follows easily from [9, Lemma] that there exists a norm-one linear projection operator $P$ on $C(K)$ such that $P(C(K))$ is separable and $Z \subseteq P(C(K))$. Hence, according to Sobczyk’s theorem, this implies that each copy of $c_0$ in $C(K)$ is complemented. This fact prevents $C(K)$ to contain a copy of $\ell_\infty$ whenever $K$ is Valdivia compact. However, a compact topological space $K$ may contain nontrivial convergent sequences and still $C(K)$ to contain a copy of $\ell_\infty$. For instance, if $Q$ denotes the weak* dual ball of $\ell_\infty$, then $\ell_\infty$ is embedded in $C(Q)$ and $Q$ contains nontrivial weak* null sequences.

2. Complemented copies of $c_0$ in $C_0(\Omega)$

Let us consider a wide class of locally compact topological spaces $\Omega$ such that $C_0(\Omega)$ contains a complemented copy of $c_0$.

Theorem 5 (Main result) Let $\Omega$ be a locally compact topological space. If $\Omega = \bigcup_{n=1}^{\infty} U_n$, where $\{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty pairwise disjoint open sets, then $C_0(\Omega)$ contains a norm-one complemented copy of $c_0$.

Proof. For each $i \in \mathbb{N}$ choose $\omega_i \in U_i$, use local compactness to select a compact neighborhood $V_i$ of $\omega_i$ in $\Omega$ such that $\omega_i \in V_i \subseteq U_i$ and pick a regular Borel probability measure $\mu$ in $\Omega$ such that $\mu(V_i) > 0$. Take for instance $\mu = \sum_{i=1}^{\infty} \frac{1}{n} \delta_{\omega_i}$ (pointwise convergence). According to Uryshon’s lemma, for each $i \in \mathbb{N}$ there is $f_i \in C_0(\Omega)$ such that $0 \leq f_i \leq 1$, $f_i(\omega) = 1$ for every $\omega \in V_i$ and $\text{supp } f_i \subseteq U_i$. Then $\{f_n\}$ is a normalized basic sequence in $C_0(\Omega)$ equivalent to the unit vector basis $\{e_n\}$ of $c_0$, with $1$ as basis constant. Actually, note that if $f = \sum_{i=1}^{\infty} a_i f_i \in [f_n]$ with $a_k \to 0$ and $\varepsilon > 0$, choosing $k \in \mathbb{N}$ such that $|a_k| < \varepsilon$ for each $i > k$ then $Q = \bigcup_{i=1}^{k} \text{supp } f_i$ is compact and $|f(\omega)| < \varepsilon$ for each $\omega \in \Omega - Q$, hence $f \in C_0(\Omega)$. Since each $f \in C_0(\Omega)$ is $\mu$-integrable, we may define the linear functionals $v_i : C_0(\Omega) \to \mathbb{R}$ in terms of the conditional expectation value of $f$ relative to the event $V_i$ by

$$v_i(f) = \text{Ev}_i(f) = \frac{1}{\mu(V_i)} \int_{V_i} f \, d\mu$$

for each $i \in \mathbb{N}$. Given $f \in C_0(\Omega)$, for each $\varepsilon > 0$ there exists a compact set $K_{f, \varepsilon} \subseteq \Omega$ such that $|f(\omega)| < \varepsilon$ for all $\omega \in \Omega - K_{f, \varepsilon}$. Since $\{K_{f, \varepsilon} \cap U_i : i \in \mathbb{N}\}$ is a covering of the compact topological subspace $K_{f, \varepsilon}$ by
open sets (in the relative topology of $K_{f,e}$) there must be a $j \in \mathbb{N}$ such that $K_{f,e} \cap U_i = \emptyset$ for each $i \geq j$. Therefore, since $V_i \subseteq \text{supp } f_i \subseteq U_i$, one has

$$|v_i(f)| = |E_{V_i}(f)| \leq \epsilon$$

whenever $i \geq j$. This establishes that $v_i(f) \to 0$. Consequently the linear operator $P : C_0(\Omega) \to [f_i]$ given by $Pf = \sum_{i=1}^{\infty} v_i(f) f_i$ is well-defined. Furthermore, since each $v_i$ is a bounded linear functional on $C_0(\Omega)$, with $|v_i(f)| \leq \|f\|_{\infty}$ for each $f \in C_0(\Omega)$, it follows immediately that $P$ is a norm-one linear operator. Finally, given that

$$Pf_j = \sum_{i=1}^{\infty} v_i(f_j) f_i = \sum_{i=1}^{\infty} E_{V_i}(f_j) f_i = f_j$$

due to the fact that the $f_i$ are disjointly supported and $f_j(\omega) = 1$ for each $\omega \in V_j$, we conclude that $P$ is a norm-one linear projection operator from $C_0(\Omega)$ onto $[f_i]$. 

**Example 1** Consider the set $\mathbb{N}$ of positive integers equipped with its discrete topology. If $\Sigma$ denotes the $\sigma$-algebra of all subsets of $\mathbb{N}$, the Stone space $\beta \mathbb{N}$ of $\Sigma$, that is, the collection of all nontrivial $\{0, 1\}$-valued additive measures defined on $\Sigma$ provided with the relative topology of the product space $\{0, 1\}^\mathbb{N}$, coincides with the Stone-Čech compactification $\beta \mathbb{N}$ of $\mathbb{N}$. If $\{A_n : n \in \mathbb{N}\}$ is a partition of $\mathbb{N}$ by nonempty sets, then $\Omega = \bigcup_{n=1}^{\infty} \hat{A}_n$, where $\hat{A}$ represents the clopen set $\mu \in S_\Sigma : \mu(A) = 1$, is a dense locally compact subspace of $\beta \mathbb{N}$ formed by a disjoint union of open sets. So, although $C(\mathbb{N})$ contains no complemented copy of $c_0$, according to the previous theorem $C_0(\Omega)$ contains a complemented copy of $c_0$. 

**Corollary 2** If $\Omega$ is a locally compact topological space such that $\Omega = \bigcup_{n=1}^{\infty} U_n$, where $\{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty pairwise disjoint open sets, then $C_0(\Omega)$ is not a Grothendieck space. 

**Proof.** Since, as a consequence of Theorem 5, $c_0$ is a quotient of $C_0(\Omega)$ via some linear bounded operator $T$, then $T^*$ embeds $\ell_1$ isometrically into $C_0(\Omega)^*$ and $\{T^* e_n\}$, where $\{e_n\}$ stands for the unit vector basis of $\ell_1$, is a weakly* null sequence in $C_0(\Omega)^*$ which is not weakly null. 

**3. Copies of $\ell_\infty$ in $C(\Omega)$**

**Theorem 6** Let $\Omega$ be a locally compact topological space. If $\Omega = \bigcup_{n=1}^{\infty} U_n$, where $\{U_n : n \in \mathbb{N}\}$ is a sequence of nonempty pairwise disjoint open sets, then $C(\Omega)$ contains a subspace linearly isometric to $\ell_\infty$.

**Proof.** It suffices to prove the theorem by assuming that $C(\Omega)$ is a real Banach space. So, if $\{\xi_n\}$ is a real bounded sequence, the function $f_\xi : \Omega \to \mathbb{R}$ defined by $f_\xi(\omega) = \xi_n$ whenever $\omega \in U_n$ is continuous on $\Omega$. Since $\Omega$ is $C^*$-embedded in $\beta \Omega$, there is a (unique) continuous extension $f_\xi^\beta$ of $f_\xi$ on $\beta \Omega$ such that $\|f_\xi^\beta\|_\infty = \|f_\xi\|_\infty = \|\xi\|_\infty$. Consequently, the mapping $\varphi : \ell_\infty \to C(\Omega)$ defined by $\varphi \xi = f_\xi^\beta$ is a linear isometry from $\ell_\infty$ into $C(\Omega)$. 

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**References**


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