On the uniform limit of quasi-continuous functions

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Abstract. We study when the uniform limit of a net of quasi-continuous functions with values in a locally convex space $X$ is a quasi-continuous function, emphasizing that this fact depends on the least cardinal of a fundamental system of neighbourhoods of $0$ in $X$, and giving necessary and sufficient conditions. The main result of the paper is Theorem 15, where the results of [7] and [10] are improved, in relation with a Theorem of L. Schwartz.

Sobre el límite uniforme de funciones quasi-continuas

Resumen. Estudiamos cuando el límite uniforme de una red de funciones quasi-continuas con valores en un espacio localmente convexo $X$ es también una función quasi-continua, resaltando que esta propiedad depende del menor cardinal de un sistema fundamental de entornos de $0$ en $X$, y estableciendo condiciones necesarias y suficientes. El principal resultado de este trabajo es el Teorema 15, en el que los resultados de [7] y [10] son mejorados, en relación al Teorema de L. Schwartz.

In general, we shall work with a measure space $(\Omega, \Sigma, \mu)$ where $\Omega$ is a topological space, the $\sigma$-algebra $\Sigma$ contains the Borel sets of $\Omega$ and $\mu(\Omega) = 1$. Suppose that $X$ is a locally convex Hausdorff space.

We say that a function $f : \Omega \to X$ is quasi-continuous if the set of points $D$ where $f$ is not continuous has outer measure $\mu^*(D) = 0$.

A function $f : \Omega \to X$ is said to be Lusin measurable if for any $\varepsilon > 0$, there is a closed set $F \subseteq \Omega$ such that $\mu(\Omega \setminus F) < \varepsilon$ and the restriction $f|_F$ is continuous.

We shall use the following axiom:

Axiom L. The interval $[0, 1]$ cannot be covered by a family $(F_i)_{i \in I}$ of closed subsets of Lebesgue measure zero where the cardinal of $I$ is less than the continuous $c$.

Then, according to [11,1-6-4], we have:

Proposition 1 Let $\Omega$ be a compact metrizable space, and let $\mu$ be a Radon measure on $\Omega$. Then Axiom L implies that the union of a family $(F_i)_{i \in I}$ consisting of closed sets of measure zero such that $\text{card}(I) < c$ does not cover any set of positive measure.

Theorem 1 Assume Axiom L in the conditions of Proposition 1. Let $X$ be a locally convex space with a base of neighbourhoods $(V_a)_{a \in A}$ of zero such that $\text{card}(A) < c$. Let $(f_i)_{i \in I}$ be a net of quasi-continuous functions $f_i : \Omega \to X$, converging uniformly to $f$. Then, if $C$ is the set of points where $f$ is continuous, we have $\mu^*(C) = 1$. 
It is obvious that if $X$ is a metrizable space, then $f$ is quasi-continuous. In the general case, we can suppose, using [8,5.4], that $X$ is the product $\Pi_{\alpha \in A} X_{\alpha}$ of a family of Banach spaces. Let $\pi_{\alpha}$ be the projection $X \rightarrow X_{\alpha}$ and $\omega_{\alpha}$ the oscillation function of $\pi_{\alpha} \circ f$; then $\pi_{\alpha} \circ f$ is quasi-continuous. Let

$$G_{\alpha,n} = \{ x \in \Omega : \omega_{\alpha}(x) < 1/n \}.$$ 

Then, since $\omega_{\alpha}$ is upper semicontinuous, $G_{\alpha,n}$ is an open set of measure $\mu(G_{\alpha,n}) = 1$. By Proposition 1, the union $\bigcup_{\alpha \in A, n \in \mathbb{N}} (X \setminus G_{\alpha,n})$ does not cover any set of positive measure. Hence, its inner measure is zero and

$$\mu^*(\bigcap_{\alpha \in A, n \in \mathbb{N}} G_{\alpha,n}) = 1.$$ 

To finish, it suffices to note that $C = \bigcap_{\alpha \in A, n \in \mathbb{N}} G_{\alpha,n}$. □

It is also useful to consider the following axiom:

**Axiom M** If $(A_i)_{i \in I}$ is a family of subsets of $[0,1]$ with Lebesgue measure zero and such that $\text{card}(I) < c$, then the union of the $A_i$’s has measure zero.

According to [11,1.6.2], we have:

**Proposition 2** Assume Axiom M and suppose that $(\Omega, \Sigma, \mu)$ is a probability space, where $\Sigma$ is countably generated. Then, if $(A_i)_{i \in I}$ is a family of subsets with measure zero and such that $\text{card}(I) < c$, we have that the union also has measure zero.

**Theorem 2** Assume Axiom M in the conditions of Proposition 2. Let $X$ be a locally convex space with a base of neighbourhoods $(Y_{\alpha})_{\alpha \in A}$ of zero such that $\text{card}(A) < c$. Let $(f_{i})_{i \in I}$ be a net of quasi-continuous functions $f_{i} : \Omega \rightarrow X$, converging uniformly to $f$. Then $f$ is quasi-continuous.

**Proof.** We just have to proceed as in Theorem 1, using Proposition 2 instead of Proposition 1. □

**Remark 1** In Theorems 1 and 2, if $\Omega$ is a $(\mathcal{T}_1)$-space and the measure $\mu$ is diffuse, then we have that $\text{card}(\Omega) \geq c > \text{card}(A)$. As we shall soon see, these theorems do not hold if $\Omega = [0,1]$, $\mu$ is the Lebesgue measure and $\text{card}(A) = c$.

**Theorem 3** Let $\Omega = [0,1]$, $\mu$ the Lebesgue measure on $\Omega$ and $X = \mathbb{R}^\mathcal{Q}$ with $\text{card}(\mathcal{Q}) = c$. Then, there exists a function $f : \Omega \rightarrow X$ which is the uniform limit of a net of quasi-continuous functions, which is not continuous at any point and having the property that any restriction $f|_H$ ($H \subseteq \Omega$) is continuous only on the countable set consisting of the isolated points of $H$. Hence, $f$ is not Luzin measurable. Moreover, for any set $H \subseteq \Omega$, there exists an open set $G \subseteq X$ such that $f^{-1}(G) = H$ and $f$ is not Borel measurable, but it is weakly measurable and its Pettis integral $\int f d\mu$ is zero.

**Proof.** We can assume that $A = \Omega$. Let $f = (X_{\alpha})_{\alpha \in A}$, where $X_{\alpha}(x) = 1$ for $x = \alpha$ and $X_{\alpha}(x) = 0$ for $x \neq \alpha$. The function $f : \Omega \rightarrow X$ is nowhere continuous but it is the uniform limit of a net $(f_{i})_{i \in I}$, where every $i \in I$ is a finite subset of $\Omega$, $I$ is ordered by inclusion and $\pi_{\alpha} \circ f_{i} = X_{\alpha}$ for every $\alpha \in i$ and $\pi_{\alpha} \circ f_{i} = 0$ for any $\alpha \notin i$, which implies that every function $f_{i}$ is quasi-continuous. Moreover, let $H$ be a subset of $\Omega$, then the restriction $f|_H$ is continuous only on the countable set that contains the isolated points of $H$.

Finally, if $U = (0,2)$ and $U_{\alpha} = \pi_{\alpha}^{-1}(U)$, we have

$$f^{-1}(U_{\alpha}) = X_{\alpha}^{-1}(U) = \alpha,$$

and hence $G = \bigcup_{\alpha \in H} U_{\alpha}$ is an open subset of $X$ such that

$$f^{-1}(G) = \bigcup_{\alpha \in H} f^{-1}(U_{\alpha}) = H.$$ 

In addition, for any $x^* \in X^*$, since by [3,Proposition 3.14.1] $x^* \circ f$ vanishes outside a finite set, it follows that $x^* \circ f$ is zero almost everywhere in $\Omega$ and hence, the Pettis integral $\int f d\mu$ is zero. □
Remark 2: We can choose $I$ to be the lattice consisting of the countable subsets of $\Omega$. In this case, for any sequence $(i_n)$ in $I$, there exists $i = \bigcup_{k \in \mathbb{N}} i_k \geq i_n$ for all $n \in \mathbb{N}$.

We can prove in a similar fashion the following theorem:

**Theorem 4**: Let $\Omega = A$ be a $(T_1)$-topological space, $\mu$ a diffuse probability measure on $\Omega$ and $X = \mathbb{R}^A$. Then there exists a function $f : \Omega \to X$ which is the uniform limit of a net of quasi-continuous functions, and having the property that any restriction $f|_H$ is continuous only on the set of isolated points of $H$, and that for every subset $H \subseteq \Omega$, there exists an open subset $G \subseteq X$ such that $f^{-1}(G) = H$. Therefore, if the set of isolated points of $\Omega$ has measure less than 1, $f$ is not quasi-continuous and if there exists a non-measurable set $H$, $f$ is neither Borel measurable nor Lusin measurable.

**Remark 3**: If $\text{card}(\Omega)$ is of measure zero, the measure of the set consisting of the isolated points of $\Omega$ has $\mu$-measure zero and there is a non-measurable set in $\Omega$. According to [5,2.5], the same happens if $\text{card}(\Omega)$ is not measurable and $\mu$ is a perfect measure. We also have that if $\mu$ is a $\tau$-additive measure, the set of isolated points of $\Omega$ has measure zero. Since the same holds for every subset $H$ of $\Omega$ and the induced measure $\mu_H$, $f$ is not Lusin measurable.

A set is said to be a $D$-set if it is the set $D(f)$ of the discontinuity points of a function $f : \Omega \to \mathbb{R}$. It is clear that any $D$-set is an $F_\sigma$-set.

It is obvious that the set which contains all the discontinuity points of a function $f = (f_\alpha)_{\alpha \in A} : \Omega \to X = \mathbb{R}^A$ is $\bigcup_{\alpha \in A} D(f_\alpha)$. Then we have:

**Theorem 5**: If $X = \mathbb{R}^A$, then for any quasi-continuous function $f_\alpha : \Omega \to \mathbb{R}$, the function $f = (f_\alpha)_{\alpha \in A} : \Omega \to X$ is quasi-continuous if and only if the union of any family $(D_\alpha)_{\alpha \in A}$ of $D$-sets of measure zero, has outer measure zero.

**Theorem 6**: Suppose that the support of $\mu$ is $\Omega$ and that $X$ is an arbitrary locally convex space having a base $(V_\alpha)_{\alpha \in A}$ of neighbourhoods of zero. Let $f : \Omega \to X$ be the uniform limit of a net of quasi-continuous functions. Then $f$ is quasi-continuous if and only if any union $\bigcup_{\alpha \in A} F_\alpha$ of closed sets of $\Omega$ having measure zero, has outer measure zero.

**Proof**: Sufficiency follows as in Theorem 1. Necessity follows from Theorem 5 taking into consideration that if $F$ is a closed set with measure zero, then $\text{ind}(F) = \emptyset$, and hence, $D(X_F) = F$. In this last step we have used the fact that $\text{supp}(\mu) = \Omega$. ■

**Remark 4**: Theorem 6 can be extended to any $\tau$-additive measure $\mu$.

If $(F_\alpha)_{\alpha \in A}$ is a family of closed sets having $\mu$-measure zero, such that
\[ \mu^*(\bigcup_{\alpha \in A} F_\alpha) > 0, \]
and $\text{card}(A)$ is of measure zero, then if we suppose that $A$ is well ordered and we set $A_\alpha = F_\alpha \setminus \bigcup_{\beta < \alpha} F_\beta$, we can easily see that $f = (X_{A_\alpha})_{\alpha \in A} : \Omega \to X = \mathbb{R}^A$ is a non-measurable Borel function which is the uniform limit of quasi-continuous functions. This result holds as well, according to [5,2.5], if $\mu$ is a perfect measure and $\text{card}(A)$ is not measurable. On the other hand, if $\Omega$ is endowed with the discrete topology, every function $f : \Omega \to X$ is continuous. But if $\mu$ is a diffuse measure, the latter is equivalent to the fact that $\text{card}(\Omega)$ does not have measure zero.

It is easily checked that if $\kappa = \kappa(\mu)$ is the least cardinal of the sets $A$ having the property that $\mu^*(\bigcup_{\alpha \in A} F_\alpha) > 0$ for a family $(F_\alpha)_{\alpha \in A}$ of closed sets of measure zero, then $\kappa$ is not the supremum of a sequence of cardinals less than $\kappa$, which is obviously less than or equal to $\text{card}(\Omega)$ if $\Omega$ is a $(T_1)$-space and $\mu$ is a diffuse measure. Axiom M implies that $\kappa(\mu) \leq \mathfrak{c}$ for the Lebesgue measure $\mu$ on $\Omega = [0, 1]$, and this in turn implies Axiom L.

We say that a cardinal is primary if it is not the supremum of a sequence of cardinals less than itself.
Corollary 1 There exists a Radon measure $\mu$ and a function $f : \Omega \rightarrow X = \mathbb{R}^k$ that is not quasi-continuous nor Borel measurable and yet it is the uniform limit of a net of quasi-continuous functions.

PROOF. It turns out from Theorem 6 by taking into consideration that, according to Haydon [2, 15.1], there exists a Radon measure $\mu$ for which the union of $\cap \alpha_1$ closed sets of measure zero may be not measurable.

In the same way as in Theorem 6 but in the same direction as Theorem 1, we can give a necessary and sufficient condition in order that $\mu^*(C) = 1$ for the set $C$ consisting of the continuity points of every function $f : \Omega \rightarrow X$ which is the uniform limit of a net of quasi-continuous functions.

Theorem 7 If $(\Omega, \Sigma, \mu)$ verifies $\mu(\bigcup_{\alpha \in A} F_\alpha) = 0$ for every family $(F_\alpha)_{\alpha \in A}$ of closed subsets of measure zero with $\text{card}(A) \leq \kappa$, then there exists a probability space $(\Omega, \Sigma', \nu)$ such that $\nu$ is an extension of $\mu$ and $\kappa(\nu) > \kappa$.

PROOF. We can assume that $\kappa \geq \aleph_0$. Let $\mathcal{H}$ be the set consisting of the unions $\bigcup_{\alpha \in A} F_\alpha$ and

$$\nu^*(E) = \inf_{H \in \mathcal{H}} \mu^*(E \setminus H) \quad (E \subseteq \Omega).$$

First of all, we are going to prove that $\nu^*$ is an outer measure. Indeed, for every $\varepsilon > 0$ and $E_n \subseteq \Omega$ there exists an $H_n \in \mathcal{H}$ ($n \in \mathbb{N}$) such that

$$\nu^*(E_n) + \varepsilon 2^{-n} > \mu^*(E_n \setminus H_n),$$

and therefore

$$\sum_n \nu^*(E_n) + \varepsilon > \sum_n \mu^*(E_n \setminus H_n) \geq \mu^*(\bigcup_n E_n \setminus \bigcup_n H_n) \geq \nu^*(\bigcup_n E_n),$$

from which it follows that

$$\nu^*(\bigcup_n E_n) \leq \sum_n \nu^*(E_n).$$

Let $(E_n)$ be a disjoint sequence in $\Sigma$ and take $M \subseteq \Omega$. Then, for every $\varepsilon > 0$, there exists an $H \in \mathcal{H}$ such that

$$\nu^*(M \cap \bigcup_n E_n) + \varepsilon > \mu^*(M \cap \bigcup_n E_n \setminus H)$$

$$= \sum_n \mu^*(M \cap E_n \setminus H)$$

$$\geq \sum_n \nu^*(M \cap E_n),$$

and therefore

$$\nu^*(M \cap \bigcup_n E_n) = \Sigma_n \nu^*(M \cap E_n).$$

From the latter, it turns out that the restriction of $\nu^*$ to the $\sigma$-algebra $\Sigma' \supseteq \Sigma$ consisting of the $\nu^*$-measurable sets, is a measure $\nu$, which is an extension of $\mu$ since $\nu(B) = \mu(B)$ for every set $B \in \Sigma$. Then, since $\nu^*(H) = 0$ for every $H \in \mathcal{H}$, it follows that $\kappa(\nu) > \kappa$.

A slight change in the previous proof allows us to prove the following theorem:

Theorem 8 If $(\Omega, \Sigma, \mu)$ has the property that $\mu(\bigcup_{\alpha \in A} F_\alpha) = 0$ for every family $(F_\alpha)_{\alpha \in A}$ of closed sets having measure zero with $\text{card}(A) < \kappa$ and $\kappa$ being a primary cardinal, then there exists a probability space $(\Omega, \Sigma', \nu)$ such that $\nu$ is an extension of $\mu$ and $\kappa(\nu) \geq \kappa$. 

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Corollary 2 From axiom $L$ it follows that for every cardinal $\kappa < c$ there exists a probability measure $\mu$ on $\Omega = [0, 1]$ which is the extension of the Lebesgue measure and such that $\kappa(\mu) > \kappa$. If $c$ is not primary, $\kappa(\mu) < c$ holds for such measures, and if $c$ is primary, we can say that $\kappa(\mu) = c$ for one of them.

Theorem 9 If $\mu$ is a regular measure and there exists a measurable function $f : \Omega \to \mathbb{R}$ such that $\mu \circ f^{-1}$ is a diffuse measure, we have that $\kappa(\mu) \leq c$.

Proof. Since $\mu$ is a regular measure, by Lusin’s theorem there exists a closed set $F \subseteq \Omega$ of positive measure such that the restriction $f|_F$ is continuous. Then, if $F_\alpha = f^{-1}(\alpha) \cap F$ for $\alpha \in \mathbb{R}$, we have that every $F_\alpha$ is a closed set of measure zero and $\mu(\bigcup_{\alpha \in A} F_\alpha) = \mu(F) > 0$.

Remark 5 According to a theorem of Zink [5, 2.2], if $\mu$ is a separable measure, there is a measurable function $f : \Omega \to [0, 1]$ such that $\mu \circ f^{-1}$ is the Lebesgue measure. Similarly, by [5, 2.1], if $\mu$ is a non-atomic measure, there exists a measurable function $f : \Omega \to [0, 1]$ such that $\mu \circ f^{-1}$ is the Lebesgue measure.

Corollary 3 If $\mu$ is a regular non-atomic measure, then $\kappa(\mu) \leq c$.

Remark 6 If $\mu$ is an atomic measure then it can trivially happen that $\mu(\bigcup_{\alpha \in A} F_\alpha) = 0$ for every family $(F_\alpha)_{\alpha \in A}$ consisting of null-measure sets.

If $\mu$ is a diffuse probability measure on a metric space $\Omega$ whose density character is non-measurable then, according to [6], $\mu$ is a non-atomic measure, and hence, $\kappa(\mu) \leq c$.

Theorem 10 If $\Omega$ is a completely regular Hausdorff space and $\mu$ is a weakly $\tau$-additive measure such that for every $x \in \Omega$ and every $\varepsilon > 0$ there exists an open neighbourhood $V$ of $x$ with $\mu(V) < \varepsilon$, then $\kappa(\mu) \leq c$.

Proof. Let $\lambda$ be the restriction of $\mu$ to the $\sigma$-algebra $\mathcal{B}_0(\Omega)$ of the Baire subsets of $\Omega$. If there were an atom $B$ of $\lambda$ then there would also exist a closed atom $F \subseteq B$ of $\lambda$. Hence, by the assumption above, for each $x \in F$ there exists an open neighbourhood $V_x \in \mathcal{B}_0(\Omega)$ such that $\lambda(V_x \cap F) \neq 0$. Let $F_x = F \setminus V_x$, then $\cap_{x \in F} F_x = \emptyset$, from which it follows (taking into account the fact that $\mu$ is a weakly $\tau$-additive) that there is a sequence $(F_{x_n})$ such that $\lambda(\cap_{n \in \mathbb{N}} F_{x_n}) = 0$, and this contradicts the fact that $\lambda(F_{x_n}) = \lambda(F) > 0$. Therefore $\lambda$ is a non-atomic measure and, according to [5, 2.1], there exists a $\lambda$-measurable function $f : \Omega \to [0, 1]$ such that $\lambda \circ f^{-1}$ is the Lebesgue measure. From this it immediately follows, as in Theorem 9, that $\kappa(\mu) \leq \kappa(\lambda) \leq c$.

Remark 7 The property that for every $x \in \Omega$ and $\varepsilon > 0$ there exists an open neighbourhood $V$ of $x$ such that $\mu(V) < \varepsilon$ is equivalent to saying $\lambda^*(\{x\}) = 0$ for every $x \in \Omega$. In general, being $\Omega$ a Hausdorff space, $\sum_{x \in \Omega} \lambda^*(\{x\}) \leq \mu(\Omega)$ and, if $\sum_{x \in \Omega} \lambda^*(\{x\}) < \mu(\Omega)$, then it follows from the remaining conditions of Theorem 10 that $\kappa(\mu) \leq c$. From this it turns out that $\kappa(\mu) \leq c$ whenever $\Omega$ is a compact infinite Hausdorff group and $\mu$ is invariant under left translations.

If $\mu$ is a diffuse measure then the function $X_\omega$ $(\omega \in \Omega)$ is Lusin measurable if and only if $\lambda^*(\{\omega\}) = 0$.

Theorem 11 If $\Omega$ is a completely regular Hausdorff space and $\mu$ is a weakly $\tau$-additive measure such that its support $S$ is not separable then $\kappa(\mu) \leq c$.

Proof. Let $\lambda$ be the restriction of $\mu$ to $\mathcal{B}_0(\Omega)$ and $H$ the closure of the countable set $\{x \in \Omega : \lambda^*(\{x\}) > 0\}$. Since $S$ is non-separable, we have $S \setminus H \neq \emptyset$ and $\mu(\Omega \setminus H) > 0$ and there exists a closed set $F \in \mathcal{B}_0(\Omega)$ with positive measure $\lambda(F) > 0$ which is disjoint from $H$. Then, by applying Theorem 10 to the induced measure $\mu_F$ it turns out that $\kappa(\mu_F) \leq \kappa(\mu_F) \leq c$. ■
Remark 8 If Ω is a compact Hausdorff space and μ is a diffuse measure with separable support, the question comes down to the case when the support is a singleton. Indeed, if \( x_n \) are the points of \( \{ x \in \Omega : \lambda^*(x) > 0 \} \) and \( \sum \lambda^*(x_n) = \mu(\Omega) \), there exists a sequence \( (F_n) \) of pairwise disjoint closed Baire sets such that \( x_n \in F_n \), and hence the probability measures \( \mu_n \) defined by
\[
\mu_n(A) = \frac{\mu(A \cap F_n)}{\lambda^*(x_n)} \quad (A \in \Sigma)
\]
have the sets \( \{x_n\} \) as supports, and \( \kappa(\mu) = \min_n \kappa(\mu_n) \).

Theorem 12 If Ω is a separable Hausdorff space and μ is a diffuse measure, then \( \kappa(\mu) \leq 2^\omega \).

Proof. Let \( D \) be a dense sequence in Ω. Then for all \( x \in \Omega \) there exists an ultrafilter \( \mathcal{U}_x \) in \( D \) which converges to \( x \), and the mapping \( x \mapsto \mathcal{U}_x \) is one-to-one. Since, according to [1], the cardinal of such ultrafilters is less than or equal to \( 2^\omega \), and \( \mu \) is a diffuse measure, it follows that \( \kappa(\mu) \leq \text{card}(\Omega) \leq 2^\omega \). ■

Remark 9 If μ is a diffuse measure and the σ-algebra of the measurable sets is countably generated, then, as in [11, 1.6.2], one can deduce that \( \kappa(\mu) \leq \omega \).

Theorem 13 If Ω is a completely regular Hausdorff space and the cardinal of the support \( S \) of the measure Σ is greater than \( 2^\omega \), then \( \kappa(\mu) \leq \omega \).

Proof. By using the usual extension \( \nu \) of μ on the Stone-Cech compactification \( \beta \Omega \) of Ω, which has the property that the induced measure \( \nu_\Omega \) coincides with μ, we can assume that Ω is a compact space. Now, since \( \text{card}(S) > 2^\omega \), from the proof of Theorem 12 it follows that \( S \) is not separable, and from Theorem 11 it turns out that \( \kappa(\mu) \leq \omega \). ■

Corollary 4 If Ω is a completely regular Hausdorff space and μ is a diffuse measure such that its support has positive measure, then \( \kappa(\mu) \leq \omega \).

Theorem 14 For every cardinal \( \kappa \) there exists a diffuse measure \( \mu \) on a \((\,\mathbb{C}\,\)space Ω such that \( \kappa(\mu) > \kappa \) and \( \text{supp} \mu = \Omega \).

Proof. We may assume that \( \kappa \) is infinite. Let Ω be a set whose cardinal is greater than \( \kappa \), and let us endow Ω with the topology whose closed sets are \( \Omega \) and all the sets with cardinal less than or equal to \( \kappa \). Let \( \Sigma \) be the corresponding Borel σ-algebra on Ω, and let us define the measure \( \mu \) by putting, for \( A \in \Sigma \), either \( \mu(A) = 0 \) or \( \mu(A) = 1 \) depending on whether \( \text{card}(A) \leq \kappa \) or \( \text{card}(A) > \kappa \). Then, as \( \kappa^2 = \kappa \), it follows that \( \kappa < \kappa(\mu) \leq \text{card}(\Omega) \). ■

Theorem 15 For every cardinal \( \kappa \) there exists a completely regular space \( \Omega = (\mathbb{C}(K), \text{weak}) \) and a probability measure \( \mu \) with empty support on \( \Omega \) such that \( \kappa(\mu) > \kappa \).

Proof. We shall proceed as in [10] and [7]. We may assume that \( \kappa > \aleph_0 \) is not a limit cardinal. Let \( \omega \) be the first ordinal with cardinal \( \kappa \) and let \( T_0 = \{ \alpha : \alpha \leq \omega \} \). Let us endow \( T_0 \) with the topology consisting of all the subsets of \( T = T_0 \setminus \{ \omega \} \) and such that the neighbourhoods of \( \omega \) are the complements of the subsets of \( T \) whose cardinal are less than \( \kappa \). With this topology \( T_0 \) is a space \( (T_\alpha) \). Let \( K \) be the Stone-Cech compactification of \( T_0 \) and put \( \Omega = (\mathbb{C}(K), \text{weak}) \). The set \( \mathcal{V}_0 \) of all the neighbourhoods of \( \omega \) is stable with respect intersections of families with cardinal less than \( \kappa \), because \( \kappa \) is not a limit cardinal, and it admits a fundamental system \( \mathcal{V} \) consisting of open-closed neighbourhoods. Let \( F \) denote the set of all continuous functions from \( K \) to \( \{0, 1\} \) which vanish at \( \omega \). It is clear that \( F \) is a weakly closed set.

We shall construct two classes \( \mathcal{C} \) and \( \mathcal{D} \) of Borel sets in \( (F, \text{weak}) \) such that
(i) The smallest σ-algebra containing \( \mathcal{C} \) is the class \( \mathcal{B}_F \) of the Borel sets of \( (F, \text{weak}) \).
(ii) If $C \subseteq \mathcal{C}$ then either $C \subseteq \mathcal{D}$ or $F \setminus C \subseteq \mathcal{D}$.

(iii) The intersection of any family of elements in $\mathcal{D}$ with cardinal less than $\kappa$ is not empty.

(iv) For all $t \in K \setminus \{\omega\}$, the set $\{f \in F : f(t) = 1\}$ belongs to $\mathcal{D}$.

Then we can define a Borel measure $\lambda$ on $(F, \text{weak})$ by putting $\lambda(B) = 1$ whenever $B \in \mathcal{B}_F$ and $B$ contains the intersection of a family of elements of $\mathcal{D}$ with cardinal less than $\kappa$, and $\lambda(B) = 0$ otherwise. Hence for every family $(F_a)_{a \in A}$ of null-$\lambda$-measure subsets $F_a \in \mathcal{B}_F$ with $\text{card} A < \kappa$ we have $\lambda_*(\bigcup_{a \in A} F_a) = 0$. And $\lambda$ is a non-weakly-$\tau$-additive measure with empty support, because $F$ is the union of the open sets $G_t = \{f : f(t) = 0\}$ when $t \in K \setminus \{\omega\}$, $\lambda(F) = 1$, and $\lambda(G_t) = 0$. From this it follows that there exists a measure with similar properties on $\Omega$, which we shall keep denoting $\lambda$.

Let $k$ be an integer, $J$ a set, $(\mu^p_j)_{p \leq k, j \in J}$ Radon measures on $K$ with $\mu^p_j(K) = 1$, and $(a^p)_{p \leq k}$, $(b^p)_{p \leq k}$ rational numbers such that $a^p < b^p$. Now we define $\mathcal{C}$ to be the class of all sets $C = \bigcup_{j \in J} U_j$, where

$$U_j = \{f \in F : \forall p \leq k, \mu^p_j(f) \in (a^p, b^p)\},$$

and $k$, $J$, the measures $\mu^p_j$, $a^p$, and $b^p$ vary.

The class $\mathcal{D}$ consists of all the sets $C \subseteq \mathcal{C}$ such that for all $V \subseteq K$ there exist $j \in J$ and $f \in U_j$ with $f = 1$ on $K \setminus V$, and also of all the complements $F \setminus C$ of the sets which do not satisfy this condition. Only (iii) needs to be proved. To this end, it is enough to show that for every family $(C_a)_{a \in A} \subseteq \mathcal{C} \cap \mathcal{D}$, where $A = \{\alpha : \alpha < \alpha_0\}$ and $\alpha_0 < \omega$, and for every $V_0 \subseteq K$, there exists $f \in \bigcap_{\alpha \in A} C_\alpha$ such that $f = 1$ on $K \setminus V_0$. It is easy to prove (with the obvious notation) that for all $\alpha \in A$ there exists $\varepsilon_\alpha > 0$ such that if $W \subseteq V$ then there exist $j \in J_\alpha$ and $f \in U^{\alpha}_j$ with $f = 1$ on $K \setminus W$, where

$$U^{\alpha}_j = \{f \in F : \forall p \leq k_\alpha, \mu^p_{j, \alpha}(f) \in (a^p + \varepsilon_\alpha, b^p - \varepsilon_\alpha)\}.$$

Let $\mathcal{U}_\alpha$ be an ultrafilter on $J_\alpha$ containing all the sets

$$\{j \in J_\alpha : \exists f \in U^{\alpha}_j, f = 1 \text{ on } K \setminus V\}$$

whenever $V \subseteq K$. Let $p \leq k_\alpha$ be given, and let us put $\nu^p_{\alpha} = \lim_{\mathcal{U}_\alpha} \mu^p_{j, \alpha}$. Then, there exists $V \subseteq K$, $V \subseteq V_0$, such that $\nu^p_{\alpha}(V \setminus W) = 0$ for all $W \subseteq V$, $p \leq k_\alpha$ and $\alpha \in A$. In the same way as in [7], but performing a transfinite induction in $\alpha \in A$, it can be proved that there exist $f_\alpha$, $j_\alpha \in J_\alpha$, $V_\alpha$, $V'_\alpha \subseteq K$, and open-closed sets $H_{\alpha, 0}, H_{\alpha, 1}$ in $K$ such that

(i) The sets $V_\alpha$ satisfy $V_\alpha \subseteq \cap_{\beta < \alpha} V'_\beta$ and $V_\alpha \cap (H_{\beta_0} \cup H_{\beta_1}) = \emptyset$ for all $\beta < \alpha \in A$, with $V_1 = V$.

(ii) $\mu^{\alpha}_{j_\alpha, \alpha}(V \setminus V'_\alpha) < \varepsilon_\alpha/2$ for all $\alpha \in A$ and $p \leq k_\alpha$.

(iii) $f_\alpha \in U^{\alpha}_{j_\alpha}$ and $f_\alpha = 1$ on $K \setminus V_\alpha$.

(iv) $f_\alpha = 0$ on $V'_\alpha \subseteq V_\alpha$ and $\mu^{\alpha}_{j_\alpha, \alpha}(V'_\alpha \setminus \{\omega\}) = 0$ for all $\alpha \in A$ and $p \leq k_\alpha$.

(v) $\bar{A}_{00} = \emptyset$, $\bar{A}_{01} = K \setminus V$, $\bar{A}_{00} = \{t \in V \setminus V'_\alpha : f_\alpha(t) = 0\}$, and $\bar{A}_{01} = \{t \in V_\alpha \setminus V'_\alpha : f_\alpha(t) = 1\}$.

(vi) $\{\omega\}, H_{00}$ and $H_{01}$ are disjoint sets such that $H_{\alpha_i} \supseteq A_{\alpha_i} \cup H_{\alpha_{i-1}}$ if $\alpha$ has a predecessor $\alpha - 1$, and $H_{\alpha_i} \supseteq A_{\alpha_i} \cup \cup_{\beta < \alpha} H_{\beta_1}$ whenever $\alpha$ is a limit ordinal ($H_{00} = \emptyset$) for $i = 0, 1$.

It is obvious that for $\alpha = \alpha_0$ (or $\alpha < \alpha_0$), the open sets $G_0 = \cup_{\beta < \alpha} H_{\beta_0}$ and $G_1 = \cup_{\beta < \alpha} H_{\beta_1}$ are disjoint, and moreover

$$\overline{G_0} \cap \overline{G_1} = (G_0 \cap T) \cup (G_1 \cap T) = \emptyset,$$

because the sets $G_i \cap T$ are disjoint and open-closed in $T_0$. Then $H = \overline{G_1}$ is an open-closed set in $K$ such that $\overline{G_0} \cap H = \emptyset$ and $\omega \notin H$.
The function \( f = X_{\mathcal{H}} \) satisfies \( f = f_a \) on \( A_a = A_{a_0} \cup A_{a_1} \). Moreover, for all \( \alpha \in A \) and all \( p \leq k_\alpha \), we have \( \mu_{\alpha}^p (f \neq f_a) < \varepsilon_\alpha \). Hence it follows that \( f \in U_{\mathcal{J}_a} \) for all \( \alpha \in A \). Therefore \( f \in \cap_{a \in A} \mathcal{C}_a \) and \( f = 1 \) on \( K \setminus V_0 \).

Since for every family \((F_\alpha)_{\alpha \in A} \) of \( \lambda \)-measure sets \( F_\alpha \in \mathcal{B}_\lambda \) with \( \text{card} A < \kappa \) we have \( \lambda_\alpha (\cup_{\alpha \in A} F_\alpha) = 0 \) then, according to Theorem 8, it turns out that there exists an \( \mu \) of \( \lambda \) such that \( \kappa (\mu) \geq \kappa \).

Given a cardinal \( \kappa \), a \( \sigma \)-algebra \( \Sigma \) is said to be a \( \kappa \)-algebra provided \( \Sigma \) is stable under unions and intersections of cardinal less than \( \kappa \). A measure \( \mu \) on a \( \kappa \)-algebra \( \Sigma \) is said to be \( \kappa \)-additive provided that for every disjoint family \((H_\alpha)_{\alpha \in A} \) of sets \( H_\alpha \in \Sigma \) with \( \text{card} A < \kappa \) one has \( \mu (\cup_{\alpha \in A} H_\alpha) = \sum_{\alpha \in A} \mu (H_\alpha) \).

**Remark 10** Given a cardinal \( \kappa \), a slight modification of the above proof allows to show, without using Theorem 11, the existence of a \( \kappa \)-additive measure \( \mu \neq 0 \) taking values in \( \{0,1\} \), with empty support on a \( \kappa \)-algebra \( \Sigma \) of subsets of a space \( \Omega = (C(K), \text{weak}) \). Then \( \kappa (\mu) \geq \kappa \) holds too.

For such measures \( \mu \) taking values in \( \{0,1\} \), in a similar way and with the same notations as in the remark following Theorem 6, it turns out that if \( \text{card} A \) is non-measurable, then \( f = (X_{A_\alpha})_{\alpha \in A} : \Omega \rightarrow \mathbb{R}^\kappa \) is a non-Borel-measurable function which is the uniform limit of a net of quasi-continuous functions.

Going more deeply into this matter, we shall prove the following Theorem without the above hypothesis.

**Theorem 16** Let \( \Omega, \mathcal{D} \) and \( \kappa = \aleph_{\xi+1} \) be as in Theorem 15, let \( \Sigma \) be the \( \kappa \)-algebra generated by the Borel sets of \( \Omega \), and let \( \mu \) be the \( \kappa \)-additive measure defined on \( \Sigma \) by setting \( \mu (A) = 1 \) if \( A \in \Sigma \) and \( A \) contains an intersection of \( \aleph_\xi \) subsets of \( \mathcal{D} \), and \( \mu (A) = 0 \) otherwise. Then there exists a non-measurable union of a disjoint family of \( \kappa \) closed null measure sets \( F_\alpha \), and therefore \( f = (X_{F_\alpha})_{\alpha \in A} : \Omega \rightarrow \mathbb{R}^\kappa \) is a non-Borel-measurable function which is the uniform limit of a net of quasi-continuous functions, and \( \kappa (\mu) = \kappa \).

**Proof.** Using the same notations as in Theorem 15, let \( G_t = \{ f \in F : f(t) = 0 \} \) for \( t \in T = \{ t : t < \omega \} \). Then, in the usual way, we can obtain a disjoint family \((F_t)_{t \in T} \) of closed sets such that \( F_t \subseteq G_t \) for every \( t \) and \( \bigcup_{t \in T} F_t = \bigcup_{t \in T} G_t = F \). Assume that every union of sets \( F_t \) is \( \mu \)-measurable. Then we can define a measure \( \nu \) on all the subsets of \( T \) by setting \( \nu (H) = \mu (\bigcup_{t \in H} F_t) \) for every subset \( H \subseteq T \). Now, proceeding as in [12], we can construct a matrix \((A_t^s)\) of \( \aleph_\xi \) rows and \( \aleph_{\xi+1} \) columns whose entries are subsets of \( T \) with the following properties:

(i) For each row \( s \), \((A_t^s) \cap (A_t^{s'}) = \emptyset \) for \( t \neq t' \).

(ii) For each column \( t \), \( T \setminus \bigcup_s A_t^s \) is a set of cardinal less than \( \kappa \).

Being \( \nu (T \setminus \bigcup_s A_t^s) = 0 \), it follows that \( \nu (\bigcup_s A_t^s) = 1 \), and therefore for each \( t \) there exists \( s_t \in \mathcal{S}_t \) such that \( \nu (A_t^{s_t}) > 0 \), since the union of \( \aleph_\xi \) \( \nu \)-null measure sets has measure zero. But then there exists a row \( s \) with \( \kappa > \aleph_0 \) pairwise disjoint sets of positive \( \nu \)-measure, which contradicts the fact that the measure \( \nu \) is finite.

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**References**


On the uniform limit of quasi-continuous functions


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