RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **95** (1), 2001, pp. 57–64
Geometría y Topología / Geometry and Topology

The envelope of the Wallace-Simson lines of a triangle. A simple proof of the Steiner theorem on the deltoid.

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Abstract. A simple proof is presented of a famous, and difficult, theorem by Jakob Steiner. By means of a straightforward transformation of the triangle, the proof of the theorem is reduced to the case of the equilateral triangle. Several relations of the Steiner deltoid with the Feuerbach circle and the Morley triangle appear then as obvious.

La envolvente de las rectas de Wallace-Simson de un triángulo. Una demostración sencilla del teorema de Steiner sobre la deltoide.

Resumen. Se presenta una demostración sencilla de un teorema famoso, y difícil, de Jakob Steiner. Mediante una transformacin muy directa del triángulo, la demostración del teorema se reduce al caso del triángulo equilátero. Diversas relaciones de la deltoide de Steiner con la circunferencia de Feuerbach y con el triángulo de Morley aparecen entonces como obvias.

Introduction

In 1856 Jakob Steiner published a surprising paper (Steiner, 1856) related to a theorem obtained by Wallace in 1796 concerning a general triangle ABC. The Wallace theorem affirms that if P is any point belonging to the circle K circumscribed to the triangle ABC then the three points U, V, W, obtained by orthogonally projecting P on the three sides of the triangle are collinear.The line thus obtained is called the Wallace-Simson line of P.

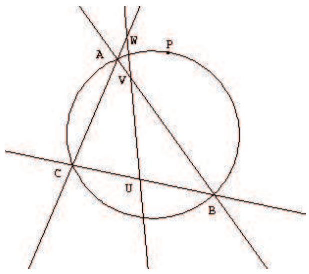

Figure 1. The Wallace line

Presentado por Alberto Dou
Recibido: 16 de Abril 2001. *Aceptado*: 9 de Mayo 2001.
Palabras clave / Keywords: Steiner deltoid, Steiner hypocicloid, Wallace-Simson lines
Mathematics Subject Classifications: 51A99
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It is rather simple to obtain a direct proof of the Wallace theorem. One has just to prove that, in Figure 1 above, one has $BVU = AVW$. Observing that quadrilateral $PVUB$ is cyclic because of the right angles at U and V one gets $BVU = BPU = 90^o - PBU = 90^o - PBC$. In the same way $PVAW$ is also cyclic and therefore

$$AVW = APW = 90^o - PAW = 90^o - (180^o - PAC) = PAC - 90^o$$

But it is quite clear that $PAC - 90^o = 90^o - PBC$, since the angles at A and B are supplementary in the cyclic quadrilateral $PBCA$ and so $BVU = AVW$. This shows that U, V, and W are collinear.

Steiner proved in his article that the envelope of such lines when P moves around the circumscribed circle is, as he states it,

> *a special curve of third class and fourth degree, which has the line at infinity as double ideal tangent, a curve that is tangent to the three sides and to the three altitudes of the triangle, and has three cuspidal points and the three tangent lines on them meet at a point.*

Along the proof of the theorem Steiner proves some other interesting properties of this interesting curve, which today is called the Steiner deltoid. Later on the curve was identified as a tricuspidal hypocycloid, $i.e.$ it is a curve described by a point on the border of a circle of radius r that rolls without sliding keeping internally tangent to another circle of radius $3r$.

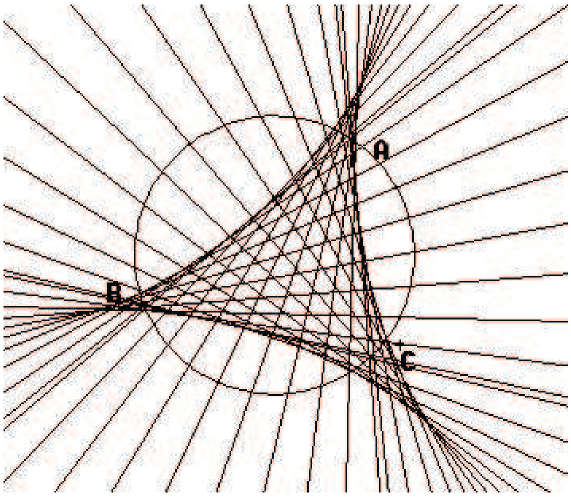

Figure 2. Steiner's deltoid

The rather surprising fact that Steiner discovered can be now easily experimented by means of any of the programs that we now have at our disposal. The pictures that are presented, in particular the one in Figure 2, have been obtained with DERIVE.

The different proofs of the theorem that have been until now offered are rather artificial and elaborate. Steiner's proof, which proceeds by reasoning directly about the original triangle ABC, is not easy to read. The proof that the curve is a tricuspidal hypocycloid is, at least the one that is presented by Heinrich Dörrie, (Dörrie, 1958), one of those proofs one would never dare to try unless one knew already the result and is not very illuminating. There have been many papers, ancient and recent, related to the Steiner deltoid. The reader is invited to look at some of the ones referred to at the end of this paper.

In this note I present, through several rather simple lemmas which only require the most basic facts of elementary geometry, a direct proof which identifies the envelope of the Wallace-Simson lines with a particular hypocycloid, determining the circles which give rise to it. In this way we can obtain at the same time many of the surprising relations of the deltoid with the different elements of the initial triangle, its Feuerbach circle, its Morley triangle and so on.

The fundamental observation for what follows is contained in the first very simple lemma

Lemma 1 *Let ABC be an arbitray triangle and K its circumscribed circle. We construct the triangle $A'B'C'$ as Figure 1 shows, inscribed in the same circle K, in such a way that A coincides with A' and with $B'C'$ parallel to BC.*

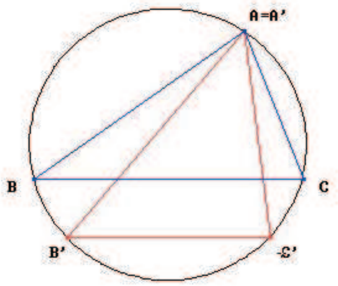

Figure 3. The key transformation

Let d be the distance between the two sides BC and $B'C'$. Let us call v the vector perpendicular to BC and of length d such that the parallel translation determined by v carries the line BC to the line $B'C'$. Then one has:

If P is an arbitrary point of K, then its Wallace-Simson line with respect to $A'B'C'$ is obtained by means of a translation determined by v from the Wallace-Simson line of P with respect to ABC.

(*Therefore the envelope of the Wallace- Simson lines of $A'B'C'$ is obtained by the parallel translation determined by v from the envelope of the Wallace-Simson lines of ABC.*)

PROOF. In the figure above QS determines the Wallace-Simson line w of P with respect to ABC and $Q'S'$ the W-S line w' of P with respect to $A'B'C'$. Since $PQSB$ is a cyclic quadrilateral, it is clear that the

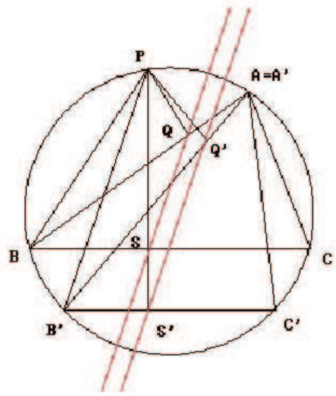

Figure 4. Transformation of the Wallace line

angles PSQ and PBQ are equal. In the same way the angle $P'Q'$ equals PBQ. Therefore w' is obtained from w by means of a parallel translation of vector v correspondig to SS'. ■

Lemma 2 *The Feuerbach circle (nine-point circle) of the triangle $A'B'C'$ is also obtained from the Feuerbach circle of ABC by the same parallel translation determined by v.*

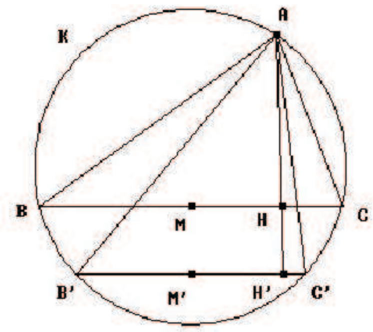

Figure 5. Transformation of the Feuerbach circle

PROOF. If R is the length of the radius of K then the length of the radius of both Feuerbach circles is $R/2$. On the other hand the Feuerbach circle F of ABC passes through M , midpoint of BC, through H, orthogonal projection of A on BC, and has its center above BC. Analogously for F', the Feuerbach circle of $A'B'C'$. This makes obvious that F' is obtained from F by a parallel translation corresponding to v. ■

Lemma 3 *The corresponding sides of the Morley triangles (the triangles determined by the trisector lines) of ABC and $A'B'C'$ are parallel.*

PROOF. According to one of the classical proofs of Morley's theorem, the one by Naraniengar (see p. 47 in Coxeter-Greitzer, *Geometry Revisited*, MAA, Washington, 1967), the side ZY of the Morley triangle close to A is at an angle $(C - B)/3$ with BC.

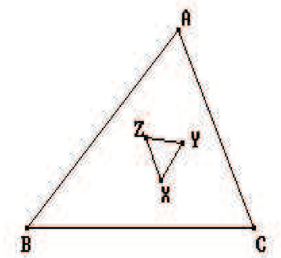

Figure 6. The Morley triangle

But one easily sees that for the two triangles ABC and $AB\prime C\prime$ one has $(C' - B')/3 = (C - B)/3$ and thus the two Morley triangles of ABC and $A'B'C'$ are as in the statement of the lemma. ■

Remark 1 When $B'C'$ gives rise to a degenerate triangle (because the line $B'C'$ passes through A or because B' coincides with C') and when $B'C'$ stays on the other side of A than BC, the above is valid if we define by continuity the different elements of the statement of the lemma in the cases of degeneration and we take $A'B'C'$ with the same orientation as ABC.

Let us call $t(A, v)$ the above defined transformation that carries the triangle ABC to $A'B'C'$, i.e. the one which keeps fixed the circumscribed circle K, that maintains $A = A'$ and translates BC to $B'C'$ by the translation determined by the vector v. Then we can state the following lemma.

Lemma 4 *Let ABC be an arbitrary triangle. Then there exist a transformation $t(A, v)$ that transforms ABC into $A'B'C'$ and another one $t(B', v')$ that transforms $B'A'C'$ into an equilateral triangle $B''A''C''$. This equilateral triangle has its sides parallel to those of the Morley triangle M of ABC and its orientation is inverse to that of M.*

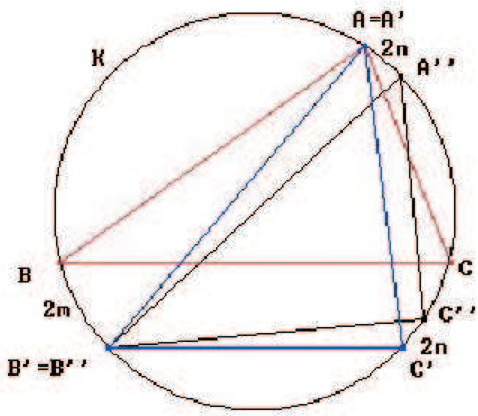

Figure 7. Two transformations make the triangle equilateral

It is enough to observe, according to Figure 7, the values of the angles of the three triangles ABC, $A'B'C'$, $A''B''C''$. If $2m$ and $2n$ are the values of the indicated arcs corresponding to the two successive transformations, then those angles are:

$$A, B, C$$
$$A' = A - 2m, B' = B + m, C' = C + m$$
$$A'' = A - 2m + n, B'' = B + m - 2n, C'' = C + m + n$$

It is now easy to see that we can choose m and n such that $A'' = B'' = C'' = 60^o$. For that it is enough to take

$$m = (180 - B - 2C)/3, n = (B - C)/3).$$

Then the value of the angle between $B''C''$ and BC is exactly $(C - B)/3$. Thus $A''B''C''$ has then its sides parallel to those of the Morley triangle of ABC. ■

Lemma 5 *For an equilateral triangle ABC the envelope of its Wallace-Simson lines is a tricuspidal hypocycloid whose vertices are the vertices of an equilateral triangle concentric with ABC, with sides parallel to those of ABC and whose size is $3/2$ that of ABC.*

PROOF. The proof can be easily obtained by coordinate geometry considerations, but we would like to stick to the methods of synthetic geometry as follows:

(a) If a rectangle with diagonal OP (diagonal length $2m$) is enlarged with two rectangles as the figure 8 shows, then one has the indicated relations between the angles.

(b) To the elements of Figure 8 one has added in Figure 9 the circles U and V, with center at O and radii $2m$ and $3m$, the equilateral triangle ABC, the circle W with center at P and radius m, that intersects QJ at T and L. Then, since the angle PLT is $3t/2$ (according to Figure 8), we have that the angle SPT is $3t$ and thus T is a point of the hypocycloid generated by W when it rolls inside V starting from the position in which the center of W is on the line OA.

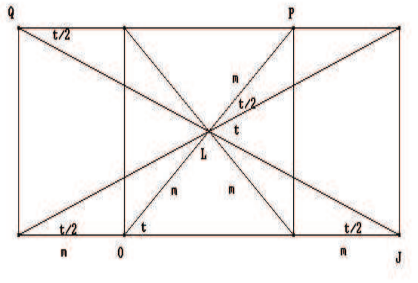

Figure 8. Preparing the way

Since the angle STQ is 90^o and ST is the instantaneous rotation radius of the circle W when it rolls inside V, the line TQ is tangent at T to the hypocycloid. On the other hand since angle PQT is $t/2$ we have (as indicated below, see Figure 10) that QT is the Wallace-Simson line of P with respect to ABC. Thus the Wallace-Simson line of P with respect to ABC is tangent to the hypocycloid at the point T. This proves the lemma. ■

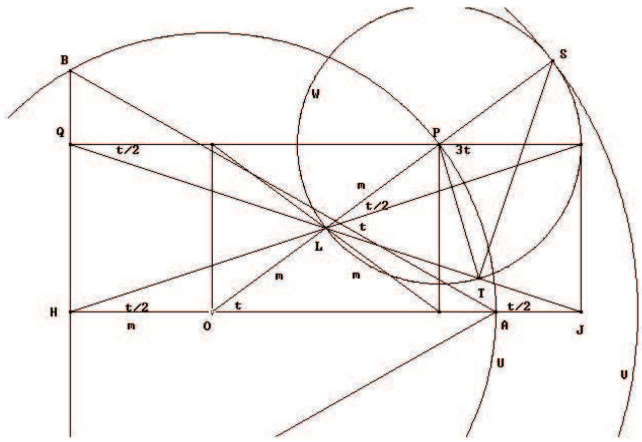

Figure 9. A synthetic proof of Steiner's theorem for the equilateral triangle

Since $BPQR$ is a cyclic quadrilateral, the angles PBR and PQR are equal. On the other hand the angle PBR, inscribed in the circle U, is half the angle POA. Thus the Wallace-Simson line of P, i.e. QR, is at an angle $t/2$ with PQ and so R coincides with the point T of the previous Figure 9.

After these lemmas we can state the theorem.

Theorem 1 *Let ABC an arbitrary triangle, K its circumscribed circle, F its Feuerbach circle, and M its Morley triangle. Let R be the length of the radius of K. (The radius of F is thus $R/2$).*

Then the envelope of the Wallace-Simson lines of ABC is a tricuspidal hypocycloid D concentric with F, tangent to F and such that its three vertices are the vertices of an equilateral triangle T. The sides of T are parallel to those of M and its orientation is inverse to that of M. The circumscribed circle to T has a radius of length $3R/2$.

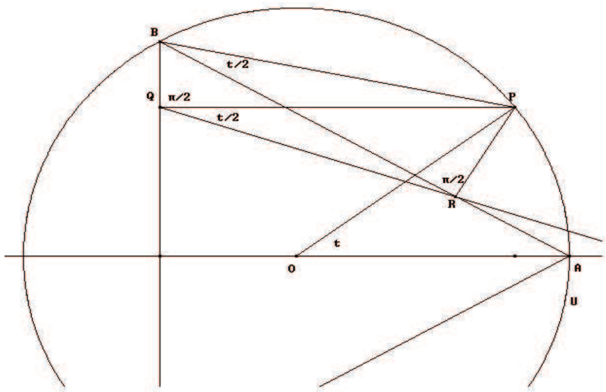

Figure 10. The Wallace line of P is tangent to the hypocycloid

(From these facts it becomes clear which are the point and the circles that give rise to D by the rolling motion.)

The statement of the theorem simply recapitulates the above lemmas and does not require any further proof.

It is also clear from this proof that the deltoid of a triangle ABC can be "determined" from the elements of ABC but its construction with only straightedge and compass is in general impossible, since it is equivalent to the trisection of a general angle.

From the Lemma 5 above one easily draws another interesting fact: Consider a point P moving on a line p with a harmonic motion of phase s. Attach to the point a straight line forming an angle $s/2$ with the line p. Then the envelope of all these lines is a deltoid.

We leave as an easy exercise the proof of this fact and the precise determination of the main elements of the corresponding deltoid.

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One can find interesting information on the web about Steiner's deltoid

There are also many papers related to Steiner's deltoid. One can see below some of them ordered by date of publication:

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