On the extension of measures

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Abstract. We give neccessary and sufficient conditions for a totally ordered by extension family \((\Omega, \Sigma_x, \mu_x)_{x \in X}\) of spaces of probability to have a measure \(\mu\) which is an extension of all the measures \(\mu_x\). As an application we study when a probability measure on \(\Omega\) has an extension defined on all the subsets of \(\Omega\).

Sobre la extensión de medidas

Resumen. Se estudian y se dan varias condiciones necesarias y suficientes para que dada una familia totalmente ordenada \((\Omega, \Sigma_x, \mu_x)_{x \in X}\) por extensión de espacios de probabilidades exista una medida \(\mu\) que sea una extensión de todas las medidas \(\mu_x\). Como aplicación de ello se estudia cuando una medida de probabilidad sobre \(\Omega\) tiene una extensión definida sobre todos los subconjuntos de \(\Omega\).

A Banach measure on a set \(\Omega\) is a finite measure \(\mu \neq 0\) on \(P(\Omega)\), the power set of \(\Omega\), such that 
\(\mu(\omega) = 0\) for every \(\omega \in \Omega\).

An Ulam measure on \(\Omega\) is a Banach measure on \(\Omega\) which takes values in the set \(\{0, 1\}\).

A cardinal \(\alpha\) is real-measurable if there exists a set \(\Omega\) whose cardinal is \(\alpha\) and such that there is a Banach measure on \(\Omega\).

A cardinal \(\alpha\) is 2-measurable if there exists a set \(\Omega\) whose cardinal is \(\alpha\) and there exists an Ulam measure on \(\Omega\).

Cardinals which are not 2-measurable are called non measurable and cardinals not real-measurable are called cardinals of zero measure.

Given a probability measure space \((\Omega, \Sigma, \mu)\), we call as usual \(\mu^*\) and \(\mu_*\) the outer and inner measures associated to \(\mu\), i.e.,
\[\mu^*(A) = \inf \{\mu(X) : A \subset X \in \Sigma\}\]and
\[\mu_*(A) = \sup \{\mu(X) : A \supset X \in \Sigma\}\].

We call \((\Omega, \Sigma_0, \mu_0)\) to a fixed probability space.

Given two probability spaces \((\Omega_1, \Sigma_1, \mu_1)\) and \((\Omega_2, \Sigma_2, \mu_2)\), when we write \(\mu_1 \subset \mu_2\) we mean that \(\mu_2\) is an extension of \(\mu_1\) and, therefore, \(\Sigma_1 \subset \Sigma_2\) and \(\mu_*^1 \geq \mu_*^2 \geq \mu_+^2 \geq \mu_+^1\).

Proposition 1 Let \((\Omega, \Sigma_x, \mu_x)_{x \in X}\) be a totally ordered by extension or inclusion family of spaces of probability such that \(\mu_0 \subset \mu_x\) for every \(x \in X\). Let us consider
\[\mu_+(A) = \sup \{\mu_+(A) : x \in X\}\]
\[ \mu^*(A) = \inf \{ \Sigma_n \tau(A_n) : \cup_n A_n \supset A \}, \]

where

\[ \tau(A) = \inf \{ \mu_x^*(A) : x \in X \} \]

and \( A \) and all of the \( A_n \) are subset of \( \Omega \). Then, the following properties are all equivalent:

(i) There exists an extension \( \lambda \) of the measures \( \mu_x \), \( x \in X \).

(ii) \( \mu_* \leq \mu^* \).

(iii) \( \mu_*(A) \leq \mu^*(A) \) for every \( A \in \mathcal{S}_\sigma \), where \( \mathcal{S} = \cup_{x \in X} \Sigma_x \).

(iv) \( \mu_*(A) = \mu^*(A) \) for every \( A \in \mathcal{S}_\sigma \).

**Proof.** First of all, let us note that \( \mu_* \) and \( \mu^* \) are, respectively, an inner measure and an outer measure (see [3]).

(i) \( \Rightarrow \) (ii). Let us suppose (i). Then, since \( \mu_x \leq \lambda \leq \lambda^* \leq \mu_x^* \) for every \( x \in X \), we get that \( \mu_* \leq \lambda \leq \lambda^* \leq \tau \) and, hence, \( \Sigma_n \mu_*(A_n) \geq \Sigma_n \lambda^*(A_n) \geq \lambda^*(A) \) if \( \cup_n A_n \supset A \). It follows now that \( \mu^* \geq \lambda^* \) and \( \mu^* \geq \mu_* \).

(ii) \( \Rightarrow \) (iii). Obvious.

(iii) \( \Rightarrow \) (iv). Indeed, if \( A \in \mathcal{S}_\sigma \), there exists a disjoint sequence \( (S_n) \) in \( \mathcal{S} \) such that \( \cup_n S_n = A \) and, hence, \( \Sigma_n \mu_*(S_n) \leq \mu_*(A) \leq \mu^*(A) \leq \Sigma_n \mu^*(S_n) \). Since \( \mu_*(S_n) = \tau(S_n) = \mu^*(S_n) \) for every \( n \in \mathbb{N} \), we get that (iv) holds.

(iv) \( \Rightarrow \) (i). Let us suppose (iv). Let \( \lambda^*(A) = \inf \{ \mu^*(H) : A \subset H \in \mathcal{S}_\sigma \} \). Then, \( \lambda^* \) is an outer measure and, for every \( \epsilon > 0 \) and for every \( E \subset \Omega \), there exists \( \mathcal{H} \in \mathcal{S}_\sigma \) such that \( \mu^*(\mathcal{H}) < \lambda^*(E) + \epsilon \) and \( E \subset \mathcal{H} \). Hence \( \lambda^*(A) \leq \lambda^*(E) + \epsilon \) and \( \lambda^*(A) < \lambda^*(E) + \epsilon \). It follows from here that every \( A \in \mathcal{S} \) is \( \lambda^* \)-measurable. Moreover, \( \lambda^*(A) = \mu^*(A) \) for every \( A \in \mathcal{S} \) and \( \mu^* \) is an extension of the measures \( \mu_x \), so we get that the restriction \( \lambda^* \) of \( \lambda^* \) on the \( \sigma \)-algebra of the \( \lambda^* \)-measurable sets is an extension of the measures \( \mu_x \). Now we can easily prove that \( \lambda^* = \mu^* \) and that \( \mu^* \) is a regular outer measure. (For every \( A \subset \Omega \), \( \mu_*(A) + \tau(\Omega \setminus A) = 1 \) and \( \lambda_*(A) + \lambda^*(\Omega \setminus A) = 1 \), hence \( \lambda_*(A) = \mu_*(A) \).

It is clear that \( \mathcal{S} \) is an algebra. In what follows, \( \Omega, \Sigma_x, \mu_x, \tau, \mu_* \), \( \mu^* \) and \( \mathcal{S} \) will mean the same as in Proposition 1.

**Proposition 2** If the restriction of \( \tau \) to \( \mathcal{S} \) is a (countably additive) measure, then \( \mu_* \leq \mu^* \).

**Proof.** Let \( \lambda^*(A) = \inf \{ \sum_n \tau(A_n) : \cup_n A_n \supset A, A_n \in \mathcal{S} \} \). Then it can be easily proved that \( \lambda^* \) is an outer measure such that every \( A \in \mathcal{S} \) is \( \lambda^* \)-measurable. On the other hand, if \( (A_n) \) is a sequence in \( \mathcal{S} \) such that \( \cup_n A_n \supset A \in \mathcal{S} \) and \( S_n = A_n \setminus \cup_{k<n} A_k \), we get

\[ \sum_n \tau(A_n) \geq \sum_n \tau(S_n \cap A) = \tau(A), \]

because the restriction of \( \tau \) to \( \mathcal{S} \) is a measure and, therefore, \( \lambda^*(A) = \tau(A) \) for every \( A \in \mathcal{S} \) and \( \lambda^* \) is an extension of the measures \( \mu_x \). It follows that the restriction of \( \lambda^* \) to the \( \sigma \)-algebra of the \( \lambda^* \)-measurable sets is an extension of the measures \( \mu_x \) and it follows from Proposition 1 that \( \mu_* \leq \mu^* \).

Now it can be easily proved, using Proposition 1, that

\[ \lambda^*(A) = \inf \{ \mu^*(H) : A \subset H \in \mathcal{S}_\sigma \}, \]

and, hence, \( \lambda^* = \mu^* \).
Proposition 3 Every set \( A \in \mathcal{S} \) is \( \mu^* \)-measurable.

PROOF. If \( A \in \Sigma_x \) and \( E \subset \Omega \) we have that \( \mu_x^*(E \cap A) + \mu_x^*(E \setminus A) = \mu_x^*(E) \), so taking limits in \( x \) we get that \( \tau(E \cap A) + \tau(E \setminus A) = \tau(E) \) for every \( A \in \mathcal{S} \) and \( E \subset \Omega \). For every \( \epsilon > 0 \) there exists a sequence \( (A_n) \) of subsets of \( \Omega \) such that \( \sum_n \tau(A_n) < \mu^*(E) + \epsilon \) and \( \bigcup_n A_n \supset E \), so we get

\[
\mu^*(E \cap A) + \mu^*(E \setminus A) \leq \sum_n \tau(A_n \cap A) + \sum_n \tau(A_n \setminus A) = \sum_n \tau(A_n) < \mu^*(E) + \epsilon,
\]

so,

\[
\mu^*(E \cap A) + \mu^*(E \setminus A) = \mu^*(E),
\]

and, therefore, every \( A \in \mathcal{S} \) is \( \mu^* \)-measurable. \( \blacksquare \)

Proposition 4 If \( \mu^*(\Omega) = 1 \), then \( \mu_* \leq \mu^* \).

PROOF. If \( A \in \Sigma_x \), we get \( \mu^*(A) \leq \mu_x(A) \), \( \mu^*(\Omega \setminus A) \leq \mu_x(\Omega \setminus A) \) and

\[
1 \leq \mu^*(A) + \mu^*(\Omega \setminus A) \leq \mu_x(A) + \mu_x(\Omega \setminus A) = 1,
\]

so, \( \mu^*(A) = \mu_x(A) \) and \( \mu^* \) is an extension of the measures \( \mu_x \). Now, Proposition 3 tells us that every \( A \in \mathcal{S} \) is \( \mu^* \)-measurable, so we get that the restriction \( \lambda \) of \( \mu^* \) to the \( \sigma \)-algebra of the \( \mu^* \)-measurable sets is an extension of the measures \( \mu_x \), so Proposition 1 implies that \( \mu_* \leq \mu^* \). \( \blacksquare \)

Proposition 5 If \( \mu_* \) is \( \mu^* \)-continuous, that is, if for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \mu^*(A) < \delta \) implies \( \mu_*(A) < \epsilon \), then \( \mu_* \leq \mu^* \).

PROOF. Let \( (A_n) \) be a sequence in \( \mathcal{S} \) such that \( (A_n) \searrow \emptyset \). Then, since every \( A_n \) is \( \mu^* \)-measurable according to Proposition 3, we get that \( \mu^*(A_n) \to 0 \) and, hence, \( \tau(A_n) = \mu_*(A_n) \to 0 \), because \( \mu_* \) is \( \mu^* \)-continuous. It follows that the restriction of \( \tau \) to \( \mathcal{S} \) is a measure and it follows from Proposition 2 that \( \mu_* \leq \mu^* \). \( \blacksquare \)

Proposition 6 If \( \tau(\bigcup_n A_n) \leq \sum_n \tau(A_n) \) for any sets \( A_n \in \mathcal{S} \), then \( \mu^* = \tau \).

PROOF. Let \( \lambda^*(A) = \inf \{ \sum_n \tau(A_n) : \bigcup_n A_n \supset A, A_n \in \mathcal{S} \} \). Then, we get from our hypothesis that

\[
\lambda^*(A) = \inf \{ \tau(H) : A \subset H \in \mathcal{S}_\sigma \} \geq \tau(A)
\]

for every set \( A \subset \Omega \) and we also get that restriction of \( \tau \) to \( \mathcal{S} \) is a measure. By Proposition 2 \( \lambda^* = \mu^* \), so we finally get that \( \mu^* = \tau \). \( \blacksquare \)

Proposition 7 The following properties are equivalent:

(i) For any increasing sequence \( (A_n) \subset \mathcal{S} \) and \( A = \bigcup_n A_n \) we have \( \tau(A) = \lim_n \tau(A_n) \).

(ii) For every sequence \( A_n \searrow \emptyset \) in \( \mathcal{S}_\sigma \) we get \( \tau(A_n) \to 0 \).

(iii) \( \tau \) is \( \mu^* \)-continuous, that is, for every \( \epsilon > 0 \) there exists \( \delta > 0 \) such that \( \mu^*(A) < \delta \) implies \( \tau(A) < \epsilon \).

(iv) \( \mu^* = \tau \).

PROOF. \( (i) \Rightarrow (iv) \). Let us suppose \( i \). Let \( (A_n) \) be a sequence in \( \mathcal{S} \). Then we have

\[
\tau(\bigcup_n A_n) = \lim_n \tau(\bigcup_{k<n} A_k) \leq \sum_n \tau(A_n),
\]
so, Proposition 6 implies that $\mu^* = \tau$

(ii) $\Rightarrow$ (i). Let us suppose (ii). Let $(A_n)$ be an increasing sequence in $S$ and $A = \bigcup_n A_n$. Since

$$\tau(A_n) \leq \tau(A) \leq \tau(A_n) + \tau(A \setminus A_n),$$

it follows that $\lim_n \tau(A_n) = \tau(A)$, because $\bigcup_n A_n \in S_{\sigma}$.

(iii) $\Rightarrow$ (ii). Let us suppose (iii). Let $(A_n)$ be a decreasing sequence in $S_{\sigma}$ such that $\bigcap_n A_n = \emptyset$. Then $\mu^*(A_n) \to 0$ because, according to Proposition 3, every $A_n$ is $\mu^*$-measurable and, hence, $\tau(A_n) \to 0$.

(iv) $\Rightarrow$ (iii). It is obvious.

**Remark 1**

1. Property (iii) can not be replaced by "$\mu_\leq \leq \mu^*$ and $\mu^*(A) = 0$ implies $\tau(A) = 0$". To see this, let $\Omega = [0, 1], X = \mathbb{N}, \lambda$ the Lebesgue measure of $\Omega$, $Z = \{Z : \lambda^*(Z) = 0\}$, $\Sigma_n$ the $\sigma$-algebra generated by the interval $[\frac{k-1}{2n}, \frac{k}{2n}) \subset \Omega$ and $\mu_n$ the restriction of $\lambda$ to $\Sigma_n$. Then, $\lambda$ is an extension of the measures $\mu_n$, $\mu^* = \lambda^*$ holds and $\tau(Z) = 0$ for every $Z \in Z$. Therefore, $\mu^*(A) = 0$ implies $\tau(A) = 0$ and, yet, $\mu^* \neq \tau$, because, if $A$ is an open dense set in $\Omega$ with Lebesgue measure $\mu^*(A) < 1$ we have $\tau(A) = \mu_n^*(A) = 1$.

2. $\mu_\leq \leq \mu^*$ does not follow from "$\mu^*(A) = 0$ implies $\tau(A) = 0$". To see this, let $\Omega = \mathbb{N}, X = \mathbb{N}$, let $\Sigma_n$ be the $\sigma$-algebra generated by the subsets of $M_n = \{1, 2, \ldots, n\}$ and $(c_k)$ a sequence of $\Sigma_n$ such that $\sum_k c_k < 1$. Let $\mu_n(A) = \sum_{k \in A} c_k$ when $A \subset M_n$ and $\mu_n(A) = 1 - \sum_{k \not\in A} c_k$ when the complementary set $A^c \subset M_n$. Then, $\tau(A) = \sum_{k \in A} c_k$ when $A \subset \mathbb{N}$ is a finite set and $\tau(A) = 1 - \sum_{k \not\in A} c_k$ when $A \subset \mathbb{N}$ is an infinite set and $\mu^*(A) = \sum_{k \in A} c_k$ for all $A \subset \mathbb{N}$. Then, $\mu^*(A) = 0$ implies $\tau(A) = 0$ but $\mu^*(\mathbb{N}) < 1 = \mu_n(\mathbb{N})$. In this case $\mu^*$ is regular, since it is even a measure.

3. If, for every sequence $(x_n)$ in $X$ there exists an $x \in X$ such that $x_n \leq x$ for every $n \in \mathbb{N}$, then it follows from Proposition 7 that $\mu^* = \tau$. Moreover $\sigma(S) = S_{\sigma} = S$, when $\sigma(S)$ is the $\sigma$-algebra generated by $S$.

**Proposition 8** $\mu^*$ is a regular outer measure.

**Proof.** For every set $A \subset \Omega$ and for every $\epsilon > 0$ there exists $x \in X$ such that $\mu^*_x(A) < \tau(A) + \epsilon$. Moreover, there exists $B \in \Sigma_x$ such that $\mu^*_x(B) = \mu^*_x(A)$ and $A \subset B$, so $\tau(B) < \tau(A) + \epsilon$. Then, for every sequence $(A_n)$ of subsets of $\Omega$ there exists a sequence $(B_n)$ in $S$ such that $\tau(B_n) < \tau(A_n) + \epsilon$ and $A_n \subset B_n$. Therefore

$$\mu^*(A) = \inf \left\{ \sum_n \tau(A_n) : \cup_n A_n \supset A \right\} \geq$$

$$\geq \inf \left\{ \sum_n \tau(B_n) : \cup_n B_n \supset A, B_n \in S \right\} - \epsilon \geq$$

$$\geq \inf \left\{ \sum_n \mu^*(B_n) : \cup_n B_n \supset A, B_n \in S \right\} - \epsilon \geq$$

$$\geq \inf \left\{ \mu^*(\cup_n B_n) : \cup_n B_n \supset A, B_n \in S \right\} - \epsilon =$$

$$\geq \inf \left\{ \mu^*(H) : A \supset H \in S_{\sigma} \right\} - \epsilon \geq \mu^*(A) - \epsilon.$$

It follows immediately that

$$\mu^*(A) = \inf \left\{ \sum_n \tau(B_n) : \cup_n B_n \supset A, B_n \in S \right\} = \inf \left\{ \mu^*(H) : A \subset H \in S_{\sigma} \right\}$$

and now Proposition 3 implies that $\mu^*$ is regular. In this way we complete the result obtained in the proof of Proposition 1.

**Remark 2** It follows from this last proposition that, if $\lambda$ is the restriction of $\mu^*$ to the $\sigma$-algebra of the $\mu^*$-measurable sets, then $\mu^* = \lambda^*$, but it can happen that $\mu^* \neq \lambda^*$. 

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Proposition 9 Let us consider the following conditions:

(i) For every decreasing sequence \( (A_n) \downarrow \emptyset \) of subsets of \( \Omega \) we have

\[
\lim_n \mu^*_0(A_n) = 0.
\]

(ii) For every disjoint sequence \( (A_n) \downarrow \emptyset \) of subsets of \( \Omega \) we have

\[
\lim_n \mu^*_0(A_n) = 0.
\]

(iii) For every disjoint sequence \( (A_n) \downarrow \emptyset \) of subsets of \( \Omega \) we have

\[
\sum_n \mu^*_0(A_n) < \infty.
\]

Then, (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i).

Proof. (iii) \( \Rightarrow \) (ii). It is obvious.

(ii) \( \Rightarrow \) (i). Let \((A_n)\) be a decreasing sequence such that \( \lim_n A_n = \emptyset \) and \( \mu^*_0(A_n) > \epsilon > 0 \) for every \( n \in \mathbb{N} \). The outer regularity of \( \mu^*_0 \) implies that \( \lim_k \mu^*_0(A_n \setminus A_k) = \mu^*_0(A_n) > \epsilon \) and, therefore, for every \( n \in \mathbb{N} \) there exists \( k > n \) such that \( \mu^*_0(A_n \setminus A_k) > \epsilon \) and it is obvious that using this result we can construct a disjoint sequence \( (B_n) \) such that \( \mu^*_0(B_n) > \epsilon \) and \( \inf_n \mu^*_0(B_n) \geq \epsilon > 0 \).

Proposition 10 In the conditions of Proposition 9, we can state in Proposition 1 that \( \mu^* = \tau \) and, hence, there exists a measure \( \mu \) which is an extension of all the measures \( \mu_x \), \( x \in X \).

Proof. From condition (i) in Proposition 9, it follows that \( (A_n) \downarrow \emptyset \) in \( S_\nu \) implies that \( \tau(A_n) \to 0 \) and, hence, \( \mu^* = \tau \), according to Proposition 7.

Proposition 11 If \( (\Omega, \Sigma, \mu) \) is a maximal probability space, in the sense that every extension \( \lambda \) of \( \mu \) coincides with \( \mu \), then \( \Sigma \) is the \( \sigma \)-algebra \( \mathcal{P}(\Omega) \) of all the subsets of \( \Omega \).

Proof. If \( \mu^*(A) = 1 \), then \( \nu^*(E) = \mu^*(A \cap E) \) \( (E \subset \Omega) \) is an outer measure such that its associated measure \( \nu \) is an extension of \( \mu \) and \( A \) is a \( \nu \)-measurable set. From the maximality of \( \mu \), it follows that \( A \in \Sigma \). If \( \mu^*(A) < 1 \) and \( B \) is a \( \mu \)-measurable covering of \( A \), then \( \mu^*(A \cup (\Omega \setminus B)) = 1 \) and, therefore, \( A \cup (\Omega \setminus B) \in \Sigma \) and \( A \in \Sigma \). It follows that \( \Sigma = \mathcal{P}(\Omega) \) and the proposition is proved.

Corollary 1 If \((\Omega, \Sigma_0, \mu_0)\) verifies one of the conditions of Proposition 9, then there exists a probability space \((\Omega, \Sigma, \mu)\) which extends \((\Omega, \Sigma_0, \mu_0)\) and such that \( \Sigma \) is the \( \sigma \)-algebra \( \mathcal{P}(\Omega) \).

Proof. It follows from Zorn’s Lemma and Propositions 10 and 11: Note that if \((\Omega, \Sigma_x, \mu_x)_{x \in X}\) is a totally ordered family of extensions of \((\Omega, \Sigma_0, \mu_0)\), Proposition 10 states the existence of a measure which is an extension of all the measures \( \mu_x \). The corollary follows now from Proposition 11.

Remark 3
(1) Corollary 1 can also be proved using the Theorem of Hahn-Banach: By this theorem, there exists a finitely additive extension \( \mu \) of \( \mu_0 \) to \( \Sigma = \mathcal{P}(\Omega) \). But \( \lim_n \mu^*_0(A_n) = 0 \) for every sequence \( A_n \downarrow \emptyset \), so \( \lim_n \mu(A_n) = 0 \) and, hence, \( \mu \) is countably additive.

(2) If \( \text{Card}(\Omega) \) has zero measure and \((\Omega, \Sigma, \mu_0)\) is a fuzzy probability space, then there exists a disjoint sequence \( (A_n) \) such that \( \inf_n \mu^*_0(A_n) > 0 \).

Remark 4. Remark. The techniques used allow us to construct, using the good order of \( \mathcal{P}(\Omega) \), a well ordered family \((\Omega, \Sigma_\alpha, \mu_\alpha)_{\alpha \in A}\) (indexed by ordinal numbers) of finite and different measure spaces and another family \((\mu^*_\alpha)_{\alpha \in A}\) of outer measures such that:

(i) \( \mu_\alpha \) is the restriction of \( \mu^*_\alpha \) to the \( \sigma \)-algebra \( \Sigma_\alpha \) of the \( \mu^*_\alpha \)-measurable sets, and every \( \mu^*_\alpha \) is regular.
(ii) If $\alpha < \beta$, then $\Sigma_\alpha \subset \Sigma_\beta$ and $\mu_\alpha^* \geq \mu_\beta^*$.

(iii) The first measure $\mu_0$ is an arbitrary complete finite measure and $\mu_{\alpha+1}$ is an extension of $\mu_\alpha$.

(iv) If $\mu_\alpha^*(\Omega) = \mu_\beta^*(\Omega)$ and $\alpha < \beta$, then $\mu_\beta$ is an extension of $\mu_\alpha$.

(v) If $\mu_\alpha^*(\Omega) > \mu_\beta^*(\Omega)$ for every $\alpha < \beta$, then $\mu_\beta$ is built based on the measures $\mu_{\alpha<\beta}$ as in Proposition 1. This technique can also be used in general when $\beta$ is a limit ordinal. For the other ordinals $\beta > 0$ with a predecessor $\beta - 1$, we can use the method of Proposition 11 so that $\mu_\beta^*(A) = \mu_{\beta-1}^*(A \cap E)$ with $\mu_{\beta-1}^*(E) = \mu_{\beta-1}^*(\Omega)$ and $E \notin \Sigma_{\beta-1}$, where $E$ is the first set in the given well ordering of $\mathcal{P}(\Omega)$ with such properties, in case it exists.

(vi) The family $\{\mu_\alpha\}_{\alpha \in A}$ has a last measure $\mu$ defined on $\mathcal{P}(\Omega)$.

If $(\Omega, \Sigma_\alpha, \mu_\alpha)$ is a fuzzy probability space and $\text{Card}(\Omega)$ has zero measure, then $\mu = 0$.

A measure $\mu$ is called ultracomplete if it is defined on $\mathcal{P}(\Omega)$.

It there is no extension of the Lebesgue measure $\lambda$ to all the subsets of $[0,1]$, then there is no non atomic ultracomplete probability measure $\mu$. To see this, let us suppose that there is one such measure $\mu$. Then, there exists an increasing family $(E_x)_{x \in X}$ such that $\mu(E_x) = x$. Let $Z_x = E_x \setminus \bigcup_{y < x} E_y$. Then, it can be easily proved that

$$\nu(A) = \mu(\bigcup_{x \in A} Z_x)$$

is a measure on $\mathcal{P}([0,1])$ which is an extension of the Lebesgue measure $\lambda$, because

$$\nu((a,b)) = \mu(\bigcup_{x \in (a,b)} Z_x) = \mu(E_b \setminus E_a) = b - a.$$ 

It can be easily proved that there exists an ultracomplete extension of the Lebesgue measure $\lambda$ if and only if $c = 2^\aleph_0$ is real-measurable. Let us recall that, according to a result of Ulam ([5]), it follows from the Continuum Hypothesis that $c$ has zero measure.

It follows that, if $c$ has zero measure, then the last measure $\mu$ of the previous process is a purely atomic measure and, therefore, $\mu_0$ is purely atomic if $\mu$ is an extension of $\mu_0$, as follows from:

**Proposition 12** If $\mu$ is purely atomic extension of $\mu_0$, then $\mu_0$ is also purely atomic.

**Proof.** It is enough to prove that, if $\mu_0$ is not atomic, then $\mu$ is not atomic. To see this, let us suppose that $\mu_0$ is not atomic and let $A$ be an atom of $\mu$. Then, there exists an increasing family $(A_x)_{x \in I}$, $I = [0,1]$, of $\mu_0$-measurable sets such that $\mu_0(A_x) = x$. Then, the function $f(x) = \mu(A \cap A_x)$ is a continuous function taking only the values 0 and $\mu(A) \neq 0$. The last part of the proof remains true when $\mu$ is the measure associated to $\mu^*$ with the notation of Proposition 1 or when $\mu$ is the last measure of the previous process. \[ \square \]

**Proposition 13** If there exists an purely atomic ultracomplete extension $\mu$ of the probability measure $\mu_0$, then there exists a process which finishes in a purely atomic measure $\nu$ which is an ultracomplete extension of $\mu_0$ and which has a disjoint and complete system of atoms formed by atoms of $\mu$.

**Proof.** Let $(A_n)$ be a disjoint sequence of atoms of $\mu$ such that $\bigcup_n A_n = \Omega$. We proceed by transfinite induction. Let us suppose that we have defined $(\mu_\alpha)_{\alpha < \beta}$ so that $\mu_\beta$ is an extension of $\mu_\alpha$ for every $\alpha < \beta$. If $\beta$ is a limit ordinal, it is clear that $\mu$ is an extension of $\mu_\beta$. Let us suppose that $\beta$ has a predecessor $\beta - 1$. Then, if there exists $E \notin \Sigma_{\beta-1}$ such that $\mu(E) = 1$, then $\mu_{\beta-1}^*(E) = 1$ and we take $\mu_{\beta}^*(A) = \mu_{\beta-1}^*(A \cap E) \geq \mu(A \cap E) = \mu(A)$ and, therefore, $\mu$ is an extension of $\mu_\beta$. This first part of the process, where we use the given good order of $\mathcal{P}(\Omega)$, finishes in an ordinal $\beta$ such that $\mu(\Omega) = 0$.

Since $\mu$ is an extension of $\mu_\beta$, it follows from Proposition 12 that $\mu_\beta$ is purely atomic, but it can happen that all the sets $A_n$ are not atoms of $\mu_\beta$. Let $(B_n)$ be a disjoint and complete system of atoms of $\mu_\beta$. By the previous property, we can suppose all of the $B_n$ to be union of some of the $A_k$. To see this, let $M_k = \{ h : \mu(A_h \cap B_k) \neq 0 \}$ and let $Z_k = \cup_{h \neq M_k} A_h \cap B_k$. Then $\mu(Z_k) = 0$. Therefore, if $B_k' = B_k \setminus Z_k$, then $B_k'$ is an atom of $\mu_\beta$ and we have

$$\mu(\cup_{h \in M_k} A_h \setminus B_k') = \sum_{h \in M_k} \mu(A_h \setminus A_h \cup B_k') = 0.$$
Hence, $B_k = \cup_{h \in M_k} A_h$ is also an atom of $\mu_\beta$. It can be easily proved that the sequence $(B_k)$ is disjoint and its union is $\Omega$.

If $A_{n_1} \subset B_1$, then $B_1$ is $\mu_\beta$-measurable covering of $A_{n_1}$. Therefore, if we call $E_1 = A_{n_1} \cup (\Omega \setminus B_1)$, we have $\mu_\beta(E_1) = 1$, and we can define $\mu_{\beta+1}(A) = \mu_\beta(A \cap E_1)$, so that $\mu_{\beta+1}$ is an extension of $\mu_\beta$ and
$$\mu_{\beta+1}(A_{n_1}) = \mu_\beta(B_1) \quad \text{and} \quad \mu_{\beta+1}(A_k) = \mu_\beta(A_k) = \mu_\beta(B_n)$$
when $A_k \subset B_n$ and $n > 1$. Now we can repeat the process taking $A_{n_2} \subset B_2, E_2 = A_{n_2} \cup (\Omega \setminus B_2)$ and $\mu_{\beta+2}(A) = \mu_{\beta+1}(A \cap E_2)$. Then we get that $\mu_{\beta+2}$ is an extension of $\mu_{\beta+1}$ and $\mu_{\beta+2}(A_{n_2}) = \mu_\beta(B_n)$ for every $k \leq 2$ and $\mu_{\beta+2}(A_k) = \mu_\beta^*(A_k)$ when $A_k \subset B_n$ and $n > 2$. With the same reasonings we can construct $\mu_{\beta+k}$ with the property that it is an extension of $\mu_{\beta+k-1}$ and it verifies $\mu_{\beta+k}(A_{n_k}) = \mu_\beta(B_k)$ for every $k \leq h$ and
$$\mu_{\beta+k}^*(A_k) = \mu_\beta^*(A_k) = \mu_\beta(B_n)$$
when $A_k \subset B_n$ and $n > h$. Every $A_{n_k}$ is an atom of $\mu_{\beta+k}$ and every subset $E$ of $A_{n_k}$ is $\mu_{\beta+k}$-measurable. To see this, let us note that either $\mu(E) = 0$ or $\mu(A_{n_k} \setminus E) = 0$ and, therefore, either $\mu_{\beta+k}(E) = 0$ or $\mu_{\beta+k}(A_{n_k} \setminus E) = 0$, which proves the previous statement, since $A_{n_k}$ is a $\mu_{\beta+k}$-measurable set.

This process can finish in a measure $\mu_{\beta+n}$; also, the measures $\mu_{\beta+k}$ can be equal, but both difficulties can be overcome, and, in the worst case, we can suppose that the process does not finish like that. Then, using the same notation as in Proposition 1, if $E_k \subset A_{n_k}$ and $\tau(E_k) = \tau(A_{n_k}) = \tau(B_k)$, we have
$$\tau(\cup_{k \in M} E_k) \leq \tau(\cup_{k \in M} B_k) = \mu_\beta(\cup_{k \in M} B_k) = \sum_{k \in M} \mu_\beta(B_k) = \sum_{k \in M} \tau(E_k).$$
It follows immediately that
$$\tau(\cup_{k \in M} E_k) = \sum_{k \in M} \tau(E_k)$$
and
$$\tau(\cup_{k \in N} A_{n_k}) = \sum_{k \in N} \tau(A_{n_k}) = \sum_{k \in N} \mu_\beta(B_k) = 1.$$

Moreover, if $E_k \subset A_{n_k}$ and $\tau(E_k) = 0$, then $\mu(E_k) = 0$ and, therefore,
$$\tau(\cup_{k \in M} E_k) = \mu(\cup_{k \in M} E_k) = 0.$$
Moreover, $\tau(\Omega \setminus \cup_k A_{n_k}) = 0$, because, for every $h,$
$$\mu_{\beta+h}^*(\Omega \setminus \cup_k A_{n_k}) \leq \mu_{\beta+h}^*(\Omega \setminus \cup_{k \leq h} A_{n_k}) = \mu_\beta(\Omega \setminus \cup_{k \leq h} B_k).$$
Hence, $\tau$ is an ultracomplete measure and $\mu_{\beta+\omega} = \tau$, and we can take $\nu = \tau$. ■

**Remark 5** It can be proved that the last measure $\mu_\beta$ is independent of the well order of $\mathcal{P}(\Omega)$ that we choose. Indeed, if $\mu'_\gamma$ is the analogous measure corresponding to another well order of $\mathcal{P}(\Omega)$, and we suppose that $\mu'_\gamma$ is an extension of $\mu_\alpha$ for every $\alpha < \alpha_0 (\alpha_0 \leq \beta)$ we get that $\mu'_\gamma$ is an extension of $\mu_{\alpha_0}$. This is clear if $\alpha_0$ is a limit ordinal, and if $\alpha_0$ has a predecessor $\alpha_0 - 1$, we have $\mu_{\alpha_0}^*(A) = \mu_{\alpha_0-1}^*(A \cap E)$, where $\mu(E) = 1$. Then, $\mu_{\alpha_0}^*(A) \geq \mu_{\alpha_0}^*(A \cap E) = \mu_{\alpha_0}^*(A)$ because $\mu_{\alpha_0}^*(E) = 1$ and, therefore, $\mu_{\alpha_0}^*$ is an extension of $\mu_{\alpha_0}$. It follows that $\mu'_\gamma$ is an extension of $\mu_{\alpha_0}$. Similarly $\mu_\beta$ is an extension of $\mu'_\gamma$, and hence, $\mu_\beta = \mu'_\gamma$. The same proof shows that $\Sigma_\beta$ is the $\sigma$-algebra $\Sigma$ generated by $\Sigma_0$ and the sets of null $\mu$-measure. Therefore, $\mu_\beta$ is the restriction of $\mu$ to $\Sigma$. Now it is easy to prove that if $\mu_\beta$ is different to $\mu$, then no process starting in $\mu_0$ will finish in $\mu$. So, if a process starting in $\mu_0$ finishes in $\mu$, then $\mu_\beta = \mu$. All of this remains true even if $\mu$ is not purely atomic.

**Proposition 14** If $\mu_0$ is a probability measure and there exists a process starting in $\mu_0$ that finishes in an extension $\mu$ of $\mu_0$, then every atom of $\mu_0$ is an atom of $\mu$. 197
PROOF. Let us suppose that $\Omega$ is an atom of $\mu_0$. Let $(\Omega, \Sigma_0, \mu_0)_{\alpha \leq \beta}$ be the family of probability spaces of the process and let $\Sigma'_\alpha$ be the $\sigma$-algebra of the sets $A \in \Sigma_\alpha$ with measure $\mu_\alpha(A) \in \{0, 1\}$. Then, $\Sigma'_\alpha = \Sigma_\alpha$ for every $\alpha \leq \beta$. Clearly $\Sigma'_0 = \Sigma_0$ and, if $\Sigma'_\alpha = \Sigma_\alpha$ for every $\alpha < \alpha_0$ ($\alpha_0 \leq \beta$) then $\Sigma'_\alpha = \Sigma_{\alpha_0}$. Indeed, if $\alpha_0$ has a predecessor $\alpha_0 - 1$ then $\mu^{*\alpha_0}_\alpha(A) = \mu^{*\alpha_0}_{\alpha_0 - 1}(A \cap E)$, with $\mu^{*\alpha_0}_{\alpha_0 - 1}(E) = 1$ and, therefore, $\Sigma'_\alpha = \Sigma_{\alpha_0}$ because $\mu^{*\alpha_0}_\alpha$ takes values in $\{0, 1\}$. If $\alpha_0$ is a limit ordinal then we also have $\Sigma'_\alpha = \Sigma_{\alpha_0}$ because $S = \cup_{\alpha < \alpha_0} \Sigma_\alpha \subset \Sigma_{\alpha_0}$ and $\Sigma_{\alpha_0}$ is the $\sigma$-algebra generated by $S$ (this follows from Proposition 1). Hence, $\mu = \mu^\beta_{\alpha}$ takes values in $\{0, 1\}$ and it follows that $\Omega$ is an atom of $\mu$. It is clear that the proposition follows now, because, if $A$ is a $\mu_0$-measurable set and there exists a process starting in $\mu_0$ which finishes in $\mu$, then there exists a process starting on the induced measure $\mu_{0A}$ which finishes in $\mu_A$. ■

Corollary 2. If $\mu_0$ is a purely atomic probability measure, then there exists a process starting in $\mu_0$ and finishing in an extension $\mu$ of $\mu_0$ if and only if there exists an purely atomic ultracomplete extension of $\mu_i$.

Proposition 15. Let $\mu_0$ be a probability measure, $\mu$ an ultracomplete extension of $\mu$, $\Sigma$ the $\sigma$-algebra generated by $\Sigma_0$ and the sets with null $\nu$-measure and let $\nu$ be the restriction of $\mu$ to $\Sigma$. Then there exists a process starting in $\mu_0$ and finishing in $\mu$ if and only if $\nu = \mu$.

PROOF. It is enough to use the Remark following Proposition 13. ■

Proposition 16. If $\mu_0$ is a non atomic probability measure and there exists a process finishing in a measure $\mu \neq 0$, then the cardinal $c = 2^{\aleph_0}$ is real measurable and, therefore, if follows from the Continuum Hypothesis that $\mu = 0$, according to [5].

PROOF. As we have seen in Proposition 12, $\mu$ is a non atomic measure and, therefore, there is an ultracomplete extension of the Lebesgue measure on the interval $[0, \mu(\Omega)]$ and $c$ is a real-measurable cardinal.

Proposition 17. If $\mu_0$ is a non atomic probability measure, built using only (ZF) and the axiom of choice, like the Lebesgue measures, and there exists a process starting in $\mu_0$ and finishing in $\mu$, then $\mu = 0$.

PROOF. First of all, the Continuum Hypothesis is independent of (ZF) and the axiom of choice, according to the well known result of P. J. Cohen. On the other hand, in the construction of $\mu$ and in Proposition 18 we have only used (ZF) and the axiom of choice. Then, if $\mu \neq 0$, the negation of (CH) would follow from (ZF) and the axiom of choice, in contradiction to the result Cohen. Therefore $\mu = 0$. ■

Remark 6. If $c$ is a real-measurable cardinal we can not avoid in the previous proposition the condition $\mu_0$ is built using only (ZF) and the axiom of choice. Indeed, then there exists an ultracomplete non atomic probability measure $\mu_0$ and, obviously, every process starting in $\mu_0$ finishes in $\mu = \mu_0 \neq 0$.

Proposition 18. If $\mu_0$ is a purely atomic probability measure and there exists a process finishing in $\mu$ then, for every atom $A$ of $\mu_0$ we have $\mu(A) = \mu_0(A)$ or $\mu(A) = 0$. In the first case, the induced measure $\mu_A$ is an extension of $\mu_{0A}$ and, therefore, $A$ is also an atom of $\mu$.

PROOF. We notice that there exists a process starting in the induced measure $\mu_{0A}$ and finishing in $\mu_A$, as in Proposition 14. So, it is enough to consider the case when $\Omega$ is an atom of $\mu_0$. Let $\Omega$ be an atom of $\mu_0$ and $(\mu_\alpha)_{\alpha \leq \gamma}$ the family of the measures of the process, and let us suppose that $\mu$ is not an extension of $\mu_0$. Then there exists a first ordinal $\alpha_0$ such that $\mu_{\alpha_0}$ is not an extension of $\mu_0$. It is clear that $\alpha_0$ is a limit ordinal. It follows from the proof of Proposition 14 that $\mu^{\alpha}_\alpha(E) \in \{0, 1\}$ for every $E \subset \Omega$ and $\alpha < \alpha_0$. It follows also that $\tau(E) = \inf_{\alpha < \alpha_0} \mu^{\alpha}_\alpha(E) \in \{0, 1\}$ and, therefore, $\mu^{\alpha_0}(E) \in \{0, 1\}$. Since $\mu_{\alpha_0}$ is not an extension of $\mu_0$ it follows that $\mu_{\alpha_0}(\Omega) = 0$ and $\mu(\Omega) = 0$. ■
Corollary 3 If $\mu_0$ is a purely atomic probability measure and $(A_n)$ is a complete and disjoint system of atoms of $\mu_0$ and $\{A_n : n \in M\}$ is the set of the $A_n$ which have the property that the induced measure $\mu_0|_{A_n}$ has an ultracomplete extension $\mu_n$, then there exists a process starting in $\mu_0$ and that finishes in the measure $\mu(A) = \sum_{n \in M} \mu_n(A \cap A_n)$.

**Proof.** It follows from Propositions 13, 14 and 18, taking into account that if $B_1$ and $B_2$ are two disjoint $\mu_0$-measurable sets and there exists a process starting in $\mu_0|_{B_1}$ and finishing in $\nu_1$, then there exists a process starting in $\mu_0(B_1 \cup B_2)$ and finishing in the measure $\nu(A) = \nu_1(A \cap B_1) + \nu_2(A \cap B_2)$, where $A \subset B_1 \cup B_2$. ■

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