Multivariate Probability Integral Transformation: Application to maximum likelihood estimation

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Abstract. Let \((X_1, X_2)\) be a continuous random vector with a cdf \(F\). The probability integral transformation (pit) is the univariate random variable \(P_2 = F(X_1, X_2)\). The expression of its cdf and a simulation algorithm in terms of the quantile function given by Chakak et al [2000], when the distribution is absolutely continuous, are extended for distributions that may present singularity. Maximum likelihood estimation of the dependence parameter based on the pit is investigated by simulation. It is shown to perform well for singular families of distributions. Extension to higher dimensions is considered.

Transformación integral de distribución multidimensional: Aplicación a la estimación de máxima verosimilitud

Resumen. Sea \((X_1, X_2)\) un vector aleatorio cuya función de distribución \(F\). La pit es la variable aleatoria unidimensional \(P_2 = F(X_1, X_2)\). La expresión de su F.D., y un algoritmo de simulación en términos de la función cuantil, derda por Chakak et al [2000], cuando la distribución es absolutamente continua, son extendidas a distribuciones qui pueden tener singularidades. La estimación de máxima verosimilitud del parámetro de depenencia basada sobre la pit se hace por simutación. Esta estimación funciona bien con familias de distribuciones singulares. La extensión a grandes dimensiones es considerada.

1. Introduction

A strategy of analyzing bivariate data consists of estimating the dependence function and the marginals separately. This two step approach to stochastic modelling is often convenient because many tractable models are readily availabe for the marginal distributions. It is clearly appropriate in situations where the marginals are known, for example from previous experience, or when the marginals are not of interest.

When the structure of the underlying distribution presents some singularity such as the shock model of Marshall and Olkin [1967], parametric estimation procedures of the dependence parameters commonly using the joint density may not be valid specially when the copula is singular with a null density. A nonlinear transformation of all the univariate marginals might be useful. In the present work, we consider the pit as such.

Let \((X_1, X_2)\) be a continuous bivariate random variable with a distribution function \(F\) and \(C\) be the unique copula, which is a continuous cdf on \([0, 1]^2\), associated to \(F\) through the relation \(F(x_1, x_2) = C(F_1(x_1), F_2(x_2))\). The pit is the univariate random variable on \([0, 1]\) defined by \(P_2 = F(X_1, X_2) = \ldots\)
$C(U_1, U_2)$, which is clearly defined from copula that is invariant under strictly increasing transformation of $X_1$ and $X_2$, and hence it is free of the marginal distributions. Unlike univariate continuous random variables where the pit is uniform(0,1), the pit for a higher dimensional random vector is not uniform(0,1). For example when the marginals are independent, $K_2(p) = P[F_1(X_1)F_2(X_2) \leq p] = P[U_1U_2 \leq p] = p[1 - \log(p)]$, $p \in (0,1]$. The distribution of the pit sometimes characterizes completely the dependence structure of the copula, such as when it is Archimedean for example. Moreover, as noted in the examples herein the pit mitigates singularity; the distribution of the pit of a singular copula is a mixture of a dirac and a uniform(0,1), and that of a copula with a singular component is absolutely continuous.

To study the effects of the reliability of the maximum likelihood estimates based on the pit, we consider three families of copulas: (i) the family of singular copulas obtained by linear combination of boundary quantile functions, (ii) the family of copulas associated to the survival bivariate exponential function of Marshall and Olkin [1967] which has the loss of memory property, and (iii) the Clayton [1978] family of copulas which is a proportional hazards and a frailty model.

Recall that an absolutely continuous distribution has a positive density a.e., a singular distribution has a null density a.e., and a distribution with a singular component is an absolutely continuous distribution whose pdf does not integrate to 1.

In section 2 we give some useful properties of quantile functions for bivariate copulas that may present singularity. We consider the cdf of the pit and note its validity in the presence of singularity. An alternative whose pdf does not integrate to 1.

Let $X$ be a univariate continuous random variable distributed as $F(x)$, the quantile function is $Q(p) = F^{-1}(p) = \inf\{x : F(x) \geq p\}$. For a copula $C$, we consider sections $S_2(p) = \{(u, v) : C(u, v) = p\}$, $0 < p \leq 1$ of constant copula $C$. They are also known as distribution contours (Conway, 1979). We describe $S_2(p)$ by a function $\psi(p, u)$ defined on the interval $[p, 1]$ as $\psi(p, u) = \inf\{v : C(u, v) \geq p\}$. Since $C$ continuous

$$\psi(p, u) = \inf\{v : C(u, v) = p\}, 0 < p \leq u \leq 1,$$

also called quantile function. The quantile function $\psi(p, u)$ is simply the inverse of the increasing function $C(u, \cdot)$. The graph of $\psi(p, u)$ may be distinct from $S_2(p)$. For example when $C(u, v) = \min\{u, v\}$, $S_2(p) = \{(u, v) : u \leq p, \text{ or } v \leq p\}$, the quantile function $\psi(p, u) = p$. That is, $C(u, v) = p \iff \psi(p, u) = p$. That is, $C(u, v) = p \iff \psi(p, u) = v$, $(u, v) \in [p, 1]^2$, $p \in (0,1]$. The quantile function for the cdf $F$ can be obtained from that of the copula as $F_2(x_2) = \psi(p, F_1(x_1))$.
Proposition 1 A copula $C$ and its quantile function $\psi(\cdot)$ satisfy:

1. $C(u, \psi(p, u)) = p$ and $\psi(C(u, v), u) \leq v$,
2. $C(u, v) \leq p \iff v \leq \psi(p, u), \ p \leq u, \forall p \in [0, 1]$.

Proof. Since $C$ is continuous $C(u, \psi(p, u)) = C(u, \inf \{v : C(u, v) = p\}) = p$, and $\psi(C(u, v), u) = \inf \{w : C(u, w) = C(u, v)\} \leq v$, which completes (1). Finally $C(u, v) \leq p$, using $p = C(u, \psi(p, u))$, is equivalent to $v \leq \psi(p, u)$, as $C$ is increasing.

Examples

1. The family of symmetric singular copulas derived from the convex combination, in a rotated axes, of the upper and lower Fréchet bound copulas:

$$C_\lambda(u, v) = \max\{-\frac{1 - \lambda}{\lambda + 1} u + v + \frac{1 - \lambda}{\lambda + 1} u \vee v, 0\}, \ \lambda \in [0, 1],$$

where $u \land v = \min\{u, v\}, \ u \lor v = \max\{u, v\}$. This family of singular copulas attains the Fréchet bound copulas (lower for $\lambda = 0$, and upper for $\lambda = 1$). Note that the copula (2) is not a convex combination of the Fréchet bound copulas. For $\lambda \in (0, 1)$, let $u_0$ be such that $C(u_0, u_0) = p$, i.e. $u_0 = \frac{1}{2}\left[p + 1 + \lambda(p - 1)\right]$. The quantile function is

$$\psi_\lambda(p, u) = \begin{cases} (p - u) \frac{1 + \lambda}{1 - \lambda} + 1, & p \leq u \leq u_0 \\ p + \frac{1 - \lambda}{1 - \lambda} (1 - u), & u_0 \leq u \leq 1. \end{cases}$$

Recall that $C_0(u, v) = \max\{-1 + u + v, 0\}$ with a quantile function $\psi(p, u) = p + 1 - u, \ p \leq u \leq 1$, and $C_1(u, v) = u \lor v$, with a quantile function $\psi(p, u) = p, \ p \leq u \leq 1$.

2. The family of copulas associated to the survival bivariate exponential distribution of Marshall \& Olkin [1967] (BVE) is

$$C_\lambda(u, v) = u + v - 1 + \min\{(1 - u)(1 - v)^{1-\lambda}, (1 - u)^{1-\lambda}(1 - v)\}, \ \lambda \in [0, 1].$$

The only absolutely continuous copula in this family is $C_0(u, v) = uv$ with a quantile function $\psi_0(p, u) = \frac{p}{u}$. Except $\lambda = 0$, $C_\lambda$ has a singular component. When $\lambda \in (0, 1)$, let $u_0$ be such that $C_\lambda(u_0, u_0) = p$, i.e. $u_0$ is the solution of

$$2u_0 + (1 - u_0)^{2-\lambda} = 1 + p.$$ (4)

The quantile function is $\psi_\lambda(p, u) = 1 - \frac{u^{\frac{1}{1-\lambda}} - p^{\frac{1}{1-\lambda}}}{(1-\lambda)}$. There no closed form for the quantile function when $u \in (u_0, 1)$ (Conway [1979]). The copula is symmetric, hence the quantile function has a symmetric graph about the diagonal $v = u$.

3. Let $\phi$ be a decreasing convex function on $(0, 1)$ satisfying $\phi(1) = 0$. It defines the Archimedean copula $C(u, v) = \frac{1}{\phi^2}[\phi(u) + \phi(v)]$, whose quantile function is $\psi(p, u) = \phi^{-1}[\phi(p) - \phi(u)]$. For the purpose of illustration, we consider the Clayton (1978) family of copulas

$$C_\lambda(u, v) = [u^{-\lambda} + v^{-\lambda} - 1]^{-1/\lambda},$$

generated by $\phi(u) = \frac{u^{\lambda-1}}{\lambda}, \ \lambda > 0$. Its quantile function is $\psi(p, u) = (p^{-\lambda} - u^{-\lambda} + 1)^{-1/\lambda}$.

For a vector $(u_1, u_2, u_3) \in [0, 1]^3$, we define the quantile function for the trivariate copula as

$$\psi(p, u_1, u_2) = \inf\{u_3 : C(u_1, u_2, u_3) = p\}, \ p \in (0, 1), \ (u_1, u_2) \in [p, 1]^2.$$ This can be obtained directly for an absolutely continuous copula $C$ as

$$\psi(p, u_1, u_2) = u_3 \iff C(u_1, u_2, u_3) = p, \ 0 < p \leq u_1, u_2 \leq 1.$$
Similar to proposition 1, we have

\[ C(u_1, u_2, u_3) \leq p \iff \psi(p, u_1, u_2) \geq u_3, \]

\[ \psi(C(u_1, u_2, u_3)) = p, \]

\[ \psi(C(u_1, u_2, u_3), u_1, u_2) \leq u_3. \]

The quantile function for the multivariate case preserves the marginals. That is, for \( s = 3 \) for example, \( \psi(p, u_1, 1) = \psi(p, u_1) \) and \( \psi(p, 1, u_2) = \psi(p, u_2) \). Other properties for multivariate quantile function are presented in Imlali et al [1999].

**Examples**

(1) The upper Fréchet bound copula is \( C(u_1, u_2, u_3) = u_1 \wedge u_2 \wedge u_3 \), with a quantile function \( \psi(p, u_1, u_2) = p \). \( p \in (0, 1) \), \( (u_1, u_2) \in [p, 1]^2 \).

(2) A natural and restrictive extension of a bivariate Archimedian copula generated by one convex function such that \( \phi(1) = 0 \), is \( C(u_1, u_2, u_3) = \phi^{-1}[\phi(u_1) + \phi(u_2) + \phi(u_3)] \), where \( (-1)^k \frac{d^k}{dp^k}(p) \geq 0, k = 1, 2, 3 \). Its quantile function is \( \psi(p, u_1, u_2) = \phi^{-1}[\phi(p) - \phi(u_1) - \phi(u_2)] \). Less restrictive extensions are generated by two convex functions \( \phi_1 \) and \( \phi_2 \) are:

\begin{align*}
C_{1,2,3}(u_1, u_2, u_3) &= \phi_2 \phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)) + \phi_2(u_3) \\
C_{2,3,1}(u_1, u_2, u_3) &= \phi_2 \phi_1^{-1}(\phi_2(u_1)) + \phi_2 \phi_1^{-1}(\phi_1(u_2) + \phi_1(u_3)) \\
C_{3,2,1}(u_1, u_2, u_3) &= \phi_2 \phi_1^{-1}(\phi_2(u_2)) + \phi_2 \phi_1^{-1}(\phi_1(u_1) + \phi_1(u_3))
\end{align*}

where \( C_{ij,k}, i \neq j \neq k \neq i \leq 3 \), exhibits the same dependence between \( (U_i, U_k) \) and between \( (U_j, U_k) \) but possibly a different dependence between \( (U_i, U_j) \). All these families of copulas are particular cases of Chakak and Koehler [1995]. Conditions under which, \( C_{1,2,3} \) for example, is a copula can be found in Joe [1990] and Hillali [1998]. The quantile function is \( \psi_{1,2,3}(p, u_1, u_2) = \phi_2^{-1}[\phi_2(p) - \phi_2(\phi_1^{-1}(\phi_1(u_1) + \phi_1(u_2)))], p \in (0, 1), (u_1, u_2) \in [p, 1]^2 \).

Taking \( \phi_3(u) = \frac{u^{1-\lambda_3}}{\lambda_3} \), \( i = 1, 2 \),

\[ C_{1,2,3}(u_1, u_2, u_3) = [(u_1^{-\lambda_1} + u_2^{-\lambda_1} - 1)\frac{u_3}{\lambda_3} + u_3^{-\lambda_3} - 1]^{-1/\lambda_2}, \]

and \( \psi_{1,2,3}(p, u_1, u_2) = [p^{-\lambda_2} - (u_1^{-\lambda_1} + u_2^{-\lambda_1} - 1)\frac{u_3}{\lambda_3} + 1]^{-1/\lambda_2} \).

**2.2. The pit’s distribution**

Let \( (X_1, X_2) \) be a bivariate random variable with continuous cdf \( F \). The cdf of the pit is \( K_2(p) = P[P_2 \leq p] = P[C(U_1, U_2) \leq p] = P[C(u_1, u_2) \leq p] = P[\psi(p, U_1) \leq U_2] = P(C(u_1, u_2) : \psi(p, u_1) \leq u_2 \in \mathbb{A}], \) where \( P(C(A)) = \int_A dC(u, v) \) for a measurable set \( A \).

Various authors have given expressions for \( K_2(p) \) for specific families of copulas (see Genest et al. [1993], Capérara et al. [1997], Ghoudi et al. [1998], Barbe et al. [1996]). A general formula is presented in terms of the quantile function by Chakak et al. [2000] when the copula is absolutely continuous,

\[ K_2(p) = p + \int_p^1 \frac{\partial C}{\partial u}(u, v)|_{(u,v)\rightarrow (u,\psi(p,u))}du, 0 < p \leq 1. \tag{7} \]

Using (1) and (2) of Proposition 1, formula (7) holds true for copulas with singularities as well.

Sometimes an explicit expression for the quantile function is available only on \( \{(u, v) : u \leq v\} \), such as for the BVE copula. When \( C \) is symmetric, a useful expression is

\[ K_2(p) = p + 2 \int_p^{u_0} \frac{\partial C}{\partial u}(u, v)|_{(u,v)\rightarrow (u,\psi(p,u))}du. \]
where \( u_0 \) is the solution of (4).

Using \( p \leq \psi(p, u) \leq 1 + p - u \), we have \( 2p - C(p, p) \leq K_2(p) \leq p + \int P_u(u, 1 + p - u) \, du \). These bounds are tighter than \( K_L(p) = p \) (resp. \( K_L(p) = 1 \)) which is the pit’s cdf for the upper (resp. lower) Fréchet bound copula \( C(u, v) = u \wedge v \) (resp. \( C(u, v) = \max\{-1 + u + v, 0\} \)).

**Examples**

(1) For the singular family of copulas (2), the derivative in \( u \) is

\[
\frac{\partial C_\lambda}{\partial u}(u, v) = \begin{cases} 
1, & (u, v) \in [u < v, 0 < u - (1 - v) \frac{1}{1 + \lambda}] \\
\frac{1 - \lambda}{1 + \lambda}, & (u, v) \in [v < u, 0 < v - (1 - u) \frac{1}{1 + \lambda}].
\end{cases}
\]

Application of (7) provides

\[ K_2(p) = \lambda p + (1 - \lambda), \quad 0 \leq p \leq 1, \]

with density

\[ k_2(p) = \begin{cases} 
(1 - \lambda), & \text{if } p = 0 \\
\lambda, & \text{if } 0 < p \leq 1
\end{cases} \]

Although the copula is singular, the pit’s density is not null, making maximum likelihood estimation of \( \lambda \) possible.

Note that \( K_2(p) = \lambda K_L(p) + (1 - \lambda) K_L(p) \), is the convex combination of the p.i.t.’s of the Fréchet bound copulas.

(2) The BVE family of copulas (3) has a quantile function \( \psi(p, u) = 1 - \frac{u - p}{(1 - u) \lambda - p}, \quad p \leq u \leq u_0 \), with \( u_0 \) the solution of (4). Using (8) we have

\[ K_2(p) = p + 2(u_0 - p) - 2(1 - \lambda) \int_p^{u_0} \frac{(1 - u)(-\lambda(u - p))}{1 - (1 - u)^{1 - \lambda}} \, du. \]

with density

\[ k_2(p) = \begin{cases} 
-1 + \frac{2}{2 - (2 - \lambda)(1 - u_0)^{1 - \lambda}}, & \text{if } 0 < p < 1 - (1 - \lambda)(1 - u_0)^{1 - \lambda} \\
0, & \text{if } p = 0 \\
-\lambda, & \text{if } p = 1 - (1 - \lambda)(1 - u_0)^{1 - \lambda}
\end{cases} \]

(3) For an Archimedian copula generated by a convex decreasing function \( \phi \) on \((0, 1]\) and vanishing at 1, we have \( \frac{\partial C}{\partial u}(u, v) = \frac{\phi'(u)}{\phi'(v)} \). Replacing \( v \) by \( \psi(p, u) \), we get \( \frac{\partial C}{\partial u}(u, \psi(p, u)) = \frac{\phi'(u)}{\phi'(\psi(p, u))} \), which gives \( K_2(p) = p - \frac{\phi'(\psi(p, u))}{\phi'(p)} \). This result is derived otherwise by Genest et al (1993). Its density \( \tilde{k}_2(p) = \frac{\phi(p)\phi'(p)}{\phi'(\psi(p, u))}, \quad p \in (0, 1] \).

Taking \( \phi(u) = \frac{u - 1}{\lambda} \), \( K_2(p) = p + p(1 - p^\lambda) / \lambda \), and \( k_2(p) = (1 + \frac{1}{\lambda})(1 - p^\lambda) \).

For higher dimensional copulas a similar technique is used to derive the pit’s cdf. For simplicity we consider \( s = 3 \) first. Let \((U_1, U_2, U_3)\) be a random vector distributed as the copula \( C \), and \( P_3 = C(U_1, U_2, U_3) \).

Using the definition of the multivariate quantile function we have

**Proposition 2**

\[
K_3(p) = p + \int_p^1 \psi(p, u_1) \, du_1 + \\
\int_p^1 \left\{ \int_{\psi(p, u_1)}^1 P[U_3 \leq \psi(p, u_1, u_2) \mid U_1 = u_1, U_2 = u_2] \, du_2 \right\} \, du_1.
\]
When \( c(u_1, u_2) \) exists,
\[
K_3(p) = p + \int_p^1 \psi(p, u_1) du_1 + \int_p^1 \left[ \int_0^1 \frac{\partial^2 C}{\partial u_1 \partial u_2} (u_1, u_2, \psi(p, u_1, u_2))}{c(u_1, u_2)} du_2 \right] du_1,
\]
for \( 0 < p \leq 1 \).

**Proof.** We proceed sequentially by conditioning.
\[
K_3(p) = p + \int_0^1 P[C(U_1, U_2, U_3) \leq p] du_1 = p + \int_0^1 \psi(p, u_1) du_1 + \int_p^1 \left[ \int_0^1 \frac{\partial^2 C}{\partial u_1 \partial u_2} (u_1, u_2, \psi(p, u_1, u_2))}{c(u_1, u_2)} du_2 \right] du_1.
\]

Using
\[
P[C(U_1, U_2, U_3) \leq p] = \frac{P[U_3 \leq \psi(p, u_1, u_2)/U_1 = u_1, U_2 = u_2]}{P[U_3 \leq \psi(p, u_1, u_2)/U_1 = u_1, U_2 = u_2]},
\]
and when the density \( c(u_1, u_2) \) exists,
\[
P[U_3 \leq \psi(p, u_1, u_2)/U_1 = u_1, U_2 = u_2] = \frac{\partial^2 C}{\partial u_1 \partial u_2} (u_1, u_2, \psi(p, u_1, u_2))}{c(u_1, u_2)},
\]
which completes the proof. \( \Box \)

Some expressions of the pit’s cdf for some multivariate families of copulas are given in Barbe et al. [1996].

**Remark 1** From (10) \( p + \int_0^1 \psi(p, u) du \leq K_3(p) \). Using \( \psi(p, u) \geq p \), implies \( p(1 + (1 - p)) \leq K_3(p) \).

However \( p(1 + (1 - p)) \) is the p.i.t’s cdf of the trivariate upper Fréchet bound copula \( C(u_1, u_2, u_3) = u_1 \land u_2 \land u_3 \) (example 1 below). Generally, for \( s \geq 2 \) and the marginal density \( c(u_1, ..., u_{s-1}) \) exists, (10) generalizes to
\[
K_s(p) = P[C(U_1, ..., U_s) \leq p] = p + \int_0^1 P[C(U_1, U_2, ..., U_s) \leq p/U_1 = u_1] du_1 = p + \int_0^1 \psi(p, u_1) du_1 + \int_0^1 \int_0^1 \psi(p, u_1, u_2) du_2 du_1 + ... + \int_0^1 \int_0^1 ... \int_0^1 \frac{\partial^{s-1} C}{\partial u_1 ... \partial u_{s-1}} (u_1, ..., u_{s-1}, \psi(p, u_1, ..., u_{s-1}))}{c(u_1, ..., u_{s-1})} du_1 ... du_{s-1}.
\]

Since \( \psi(p, u_1, ..., u_k) \geq p, \forall k < s \),
\[
1 - (1 - p)^{s-1} \leq K_s(p), 0 < p \leq 1,
\]
implying that
\[
\lim_{s \to \infty} K_s(p) = 1, p > 0.
\]

This means that a higher dimensional continuous random vector has a pit almost degenerate at 0.
Examples

(1) The upper Fréchet bound copula is \( C(u_1, u_2, u_3) = u_1 \wedge u_2 \wedge u_3 \) with quantile function \( \psi(p, u_1, u_2) = p \) and \( P[U_3 \leq p/U_1 = u_1, U_2 = u_2] = 0 \). Using (10) \( K_3(p) = p(2 - p), 0 < p \leq 1 \). Its density is \( k_3(p) = 2(1 - p) \). Likewise \( 1 - (1 - p)^{s-1} \) is the pit’s cdf of the \( s \)-variate upper Fréchet copula \( C(u_1, \ldots, u_s) = u_1 \wedge \ldots \wedge u_s \).

(2) The independence copula is \( C(u_1, u_2, u_3) = u_1 u_2 u_3 \). The quantile function is \( \psi(p, u_1, u_2) = \tilde{u}_{1,2}, \psi(p, u_1) = \tilde{u}_1 \). The pit’s cdf is \( K_3(p) = p[1 + \log(1/p) + \frac{1}{p}\log^2(1/p)], p \in [0, 1], \) with a density \( k_3(p) = \frac{1}{p} \log^2(p), 0 < p \leq 1 \).

(3) The trivariate Archimedian copula generated by one parameter convex function \( \phi \) satisfying \( \frac{\partial^k}{\partial u^k} \phi^{-1}(u) \geq 0, \forall k \geq 1 \) can be found in Barbe et al. [1996]. The trivariate Archimedian copula \( C_{12,3} \), for example, generated by two convex functions \( \phi_i(u) = (u^{-\lambda_i} - 1)/\lambda_i, i = 1, 2 \) provides
\[
K_3(p) = p + \int_0^1 \psi(p, u_1, u_2) \, du_1 + \frac{p^{1+\lambda_2}}{1 + \lambda_1} \int_0^1 \int_0^1 \left[ u_1^{-\lambda_1} - u_2^{-\lambda_1} - 1 \right]^{(\lambda_2+1)/\lambda_1} \times \left[ (\lambda_2 - \lambda_1) + (1 + \lambda_2)p^{1+\lambda_2}[u_1^{-\lambda_1} - u_2^{-\lambda_1} - 1]^{\lambda_2/\lambda_1} \right] \, du_2 \, du_1,
\]
where \( \psi(p, u, u_1) = (p^{\lambda_1} - u_1^{-\lambda_1} + 1)^{-1/\lambda_1} \), which does not seem to have a closed form expression.

3. Simulation

Algorithms for generating random samples from multivariate distributions are useful in Monte-Carlo studies of properties of multivariate statistical methods. When the univariate marginals are specified, the algorithm below uses the quantile of a conditional cdf. It is an immediate extension of Proposition 5 in Chakak et al. [2000].

Proposition 3

Let \( U_1 \) and \( U_2 \) be two uniform(0,1) jointly distributed as a copula \( C \). Let \( V_1 = U_1, \) and \( V_2 = \inf\{W: \frac{\partial C}{\partial w}(U_1, W) \geq U_2 \} \). Then \( (V_1, V_2) \) are jointly distributed as \( C, \) and consequently \( P_2 = C(V_1, V_2) \) is distributed as \( K_2 \).

Examples

(1) For the family (2) with \( \lambda \in (0, 1), \)
\[
\frac{\partial C_\lambda}{\partial u_1}(u_1, w) = \begin{cases} 1, & u_1 < w, 0 < u_1 - (1 - w) \frac{1-\lambda}{\lambda}, \\ \frac{1}{w}, & w < u_1, 0 < w - (1 - u) \frac{1-\lambda}{\lambda}. \end{cases}
\]
The set \( [u_1 < w, 0 < u_1 - (1 - w) \frac{1-\lambda}{\lambda}] \) is the triangle of vertices \( (0, 1), (\frac{1-\lambda}{\lambda}, \frac{1-\lambda}{\lambda}), (1, 1) \). Likewise \( [u_1 < w, 0 < u_1 - (1 - w) \frac{1-\lambda}{\lambda}] \) is the triangle of vertices \( (1, 0), (\frac{1-\lambda}{\lambda}, \frac{1-\lambda}{\lambda}), (1, 0) \). The equation \( w = 1 - \frac{1-\lambda}{\lambda} u_1 \), and \( w = \frac{1-\lambda}{\lambda} (1 - u_1) \), represents the line joining \( (0, 1) \) and \( (\frac{1-\lambda}{\lambda}, \frac{1-\lambda}{\lambda}) \), and \( (1, 0) \) respectively. There are three cases for \( u_1 \) to consider:

(i) \( u_1 \leq \frac{1-\lambda}{\lambda}, \) implies
\[
\frac{\partial C_\lambda}{\partial u_1}(u_1, w) = \begin{cases} 0, & w < 1 - \frac{1+\lambda}{1-\lambda} u_1, \\ 1, & 1 - \frac{1+\lambda}{1-\lambda} u_1 < w. \end{cases}
\]

Hence
\[
\inf\{w: \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2 \} = 1 - \frac{1+\lambda}{1-\lambda} u_1.
\]
Then \( (U_1, 1 - \frac{1 - \lambda}{1 + \lambda} U_1) \) is distributed as \( C_\lambda \) and hence
\[
P_2 = C_\lambda(U_1, 1 - \frac{1 + \lambda}{1 - \lambda} U_1) = 0.
\]

(ii) \( \frac{1 - \lambda}{1 + \lambda} \leq u_1 \), and \( u_2 \leq \frac{1 - \lambda}{1 + \lambda} \), implies \( \inf \{ w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2 \} = \frac{1 - \lambda}{1 + \lambda}(1 - u_1) \). Then \( (U_1, \frac{1 - \lambda}{1 + \lambda}(1 - U_1)) \) is distributed as \( C_\lambda \), and
\[
P_2 = C_\lambda \left( U_1, \frac{1 - \lambda}{1 + \lambda}(1 - U_1) \right) = 0.
\]

(iii) \( \frac{1 - \lambda}{1 + \lambda} \leq u_1 \), and \( u_2 \leq \frac{1 - \lambda}{1 + \lambda} \), \( \inf \{ w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2 \} = u_1 \). Then \( (U_1, U_1) \) is distributed as \( C_\lambda \), and
\[
P_2 = C_\lambda(U_1, U_1) = \frac{2U_1}{1 + \lambda} - \frac{1 - \lambda}{1 + \lambda}.
\]

When \( \lambda = 0 \), \( (U_1, 1 - U_1) \) is distributed as \( C_0 \), and hence \( P_2 = 0 \), and when \( \lambda = 1 \), \( (U_1, U_1) \) is distributed as \( C_1 \), implying \( P_2 = U_1 \).

For \( \lambda \in (0, 1) \) the algorithm goes as follows:
Generate \( U_1, U_2 \) two independent uniform[0,1].
\[
(V_1, V_2) = \begin{cases} 
(U_1, 1 - \frac{1 - \lambda}{1 + \lambda} U_1), & \text{if } U_1 \leq \frac{1 - \lambda}{1 + \lambda}, \\
(U_1, \frac{1 - \lambda}{1 + \lambda}(1 - U_1)), & \text{if } \frac{1 - \lambda}{1 + \lambda} \leq U_1, U_2 \leq \frac{1 - \lambda}{1 + \lambda}, \\
(U_1, U_1), & \text{if } \frac{1 - \lambda}{1 + \lambda} \leq U_1, \frac{1 - \lambda}{1 + \lambda} \leq U_2,
\end{cases}
\]
is distributed as (2). The random variable
\[
P_2 = \begin{cases} 
\frac{2U_1}{1 + \lambda} - \frac{1 - \lambda}{1 + \lambda} & \text{when } (U_1, U_2) \in \left[ \frac{1 - \lambda}{1 + \lambda}, 1 \right] \times \left[ \frac{1 - \lambda}{1 + \lambda}, 1 \right] \\
0 & \text{elsewhere}
\end{cases}
\]
is distributed as \( K_2(p) = \lambda p + 1 - \lambda, \) \( 0 < p \leq 1, 0 < \lambda < 1. \)

(2) For \( 0 < \lambda < 1 \), the BVE is:
\[
C_\lambda(u_1, u_2) = \begin{cases} 
u_1 + u_2 - 1 + (1 - u_2)(1 - u_1)^{1 - \lambda} & u_1 \leq u_2 \\
u_1 + u_2 - 1 + (1 - u_1)(1 - u_2)^{1 - \lambda} & u_2 \leq u_1.
\end{cases}
\]

Its first derivative is
\[
\frac{\partial C_\lambda}{\partial u_1}(u_1, u_2) = \begin{cases} 
1 - (1 - \lambda)(1 - u_2)(1 - u_1)^{1 - \lambda}, & u_1 \leq u_2 \\
1 - (1 - u_2)(1 - u_1)^{1 - \lambda}, & u_2 \leq u_1.
\end{cases}
\]

We have \( \frac{\partial C_\lambda}{\partial u_1}(u, u +) = 1 - (1 - \lambda)(1 - u)^{1 - \lambda} \) and \( \frac{\partial C_\lambda}{\partial u_1}(u, u -) = 1 - (1 - u)^{1 - \lambda} \) which shows that \( \frac{\partial C_\lambda}{\partial u_1}(u, w) \) is not continuous only at \( w = u_1 \). For fixed \( u_1 \), we consider the three situations:

(i) \( u_2 < \frac{\partial C_\lambda}{\partial u_1}(u_1, u_1+) = 1 - (1 - u_1)^{1 - \lambda} \), implying \( \inf \{ w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2 \} = 1 - (1 - u_2)^{1/(1 - \lambda)} \).

(ii) \( \frac{\partial C_\lambda}{\partial u_1}(u_1, u_1-) < u_2 \leq \frac{\partial C_\lambda}{\partial u_1}(u_1, u_1+) \), implying \( v_2 = \inf \{ w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2 \} = u_1 \).

(iii) \( u_2 \leq \frac{\partial C_\lambda}{\partial u_1}(u_1, u_1+) \), implying \( v_2 = \inf \{ w : \frac{\partial C_\lambda}{\partial u_1}(u_1, w) \geq u_2 \} = 1 - \frac{1 - u_2}{(1 - \lambda)(1 - u_1)^{1 - \lambda}} \).

Define the subsets
\[
A = \{(u_1, u_2) : u_2 < 1 - (1 - u_1)^{1 - \lambda}, 1 - (1 - u_1)^{1 - \lambda} < u_2 \leq 1 - (1 - \lambda)(1 - u_1)^{1 - \lambda}, 1 - (1 - \lambda)(1 - u_1)^{1 - \lambda} \leq u_2 \}.
\]
The algorithm goes as follows: Generate $U_1, U_2$ two independent uniform$[0,1]$

\[
(V_1, V_2) = \begin{cases} 
(U_1, 1 - (1 - U_2)^{1/(1-\lambda)}), & \text{for } (U_1, U_2) \in \mathcal{A} \\
(U_1, U_1), & \text{for } (U_1, U_2) \in \mathcal{B} \\
(U_1, 1 - \frac{1 - U_2}{1 - \lambda (1 - U_1)^{1-\lambda}}), & \text{for } (U_1, U_2) \in \mathcal{C} 
\end{cases}
\]

is distributed as $C_\lambda$. The random variable

\[
P_2 = \left\{ \begin{array}{ll} 
U_1 - (1 - U_2)^{1/(1-\lambda)} & \text{for } (U_1, U_2) \in \mathcal{A} \\
[1 - U_1]^{1-\lambda}(1 - U_2)^{1/(1-\lambda)} & \text{for } (U_1, U_2) \in \mathcal{B} \\
U_1 - \frac{1 - U_2}{1 - \lambda (1 - U_1)^{1-\lambda}} & \text{for } (U_1, U_2) \in \mathcal{C} 
\end{array} \right.
\]

is distributed as (9).

**Proposition 4** Let $U_1, U_2, U_3$ be three independent uniform$(0,1)$, and $C$ an absolutely continuous with a density $c$. Put $V_1 = U_1, V_2$ such that $\frac{\partial^2 c}{\partial u_1 \partial u_2}(V_1, V_2, 1) = U_2$, and $V_3$ satisfying $\frac{\partial^2 c}{\partial u_1 \partial u_2}(V_1, V_2, V_3) = c(U_1, U_2, 1)$. Then $V_1, V_2, V_3$ are jointly distributed as $C$, and hence $C(V_1, V_2, V_3)$ is distributed as $K_3$.

**Proof.** From proposition 3, $(V_1, V_2)$ is distributed as $C(u_1, u_2, 1)$.

\[
P_C[v_1 \leq u_2, v_2 \leq u_2, v_3 \leq v_3] = \int_{v_1}^{v_2} \int_{v_1}^{v_2} P_C[u_1 u_2](u_1, u_2, 1) \, du_1 du_2 = C(v_1, v_2, v_3).
\]

**Example.**

The trivariate Archimedean copula generated by one convex decreasing function on $[0,1]$ such that $\phi(1) = 0$ is $C(u_1, u_2, u_3) = \phi^{-1}[\phi(u_1) + \phi(u_2) - \phi(u_3)]$. The derivatives of $C$ are $\frac{\partial^2 c}{\partial u_1 \partial u_2}(u_1, u_2, u_3) = \frac{\phi^{-1}(u_1) \phi^{-1}(u_2)}{|C(u_1, u_2, u_3)|}$ and $\frac{\partial^2 c}{\partial u_1 \partial u_2}(u_1, u_2, u_3) = -\frac{\phi^{-1}(u_1) \phi^{-1}(u_2)}{|C(u_1, u_2, u_3)|}.$

The algorithm goes as follows:

1. Put $V_1 = U_1$.
2. Solve in $V_2$ $\frac{\partial^2 c}{\partial u_1 \partial u_2}(V_1, V_2, 1) = U_2$. This gives $V_2 = \phi^{-1}[\phi(W) - \phi(U_1)]$.
3. Solve in $V_3$ $\frac{\partial^2 c}{\partial u_1 \partial u_2}(V_1, V_2, V_3) = U_3 c(V_1, V_2, 1)$ gives

\[
\frac{\phi^{-1}(C(V_1, V_2, 1))]^3}{U_3 \phi^{-1}(C(V_1, V_2, 1))} = \frac{\phi^{-1}(C(V_1, V_2, V_3))}{U_3 \phi^{-1}(C(V_1, V_2, V_3))}.
\]

We illustrate this algorithm with $\phi(u) = u^{-\lambda}, \lambda > 0$, the convex decreasing function with $\phi(1) = 0$ generating the Clayton [1978] bivariate copula. The trivariate copula is $C(u_1, u_2, u_3) = u_1^{-\lambda} + u_2^{-\lambda} + u_3^{-\lambda} - 2^{-1/\lambda}$. Let $(U_1, U_2, U_3)$ be three independent uniform$(0,1)$. Then

1. (1) consider $V_1 = U_1$.
2. $V_2 = [U_1^{-\lambda} U_2^{1/(1+\lambda)} + 1 - U_1^{-\lambda}]^{-1/\lambda}$. Then $C(V_1, V_2) = U_1 U_2^{1/(1+\lambda)}$ is distributed as $K_2(p) = p[1 + \frac{p^{-\lambda}}{\lambda}]$.
3. $V_3 = [U_3^{1/(1+\lambda)} - 1] U_1^{-\lambda} U_2^{-\lambda} U_3^{1/(1+\lambda)} + 1]^{-\frac{1}{1+\lambda}}$. Then $(V_1, V_2, V_3)$ is distributed as $[\phi^{-\lambda} + \phi^{-\lambda} + \phi^{-\lambda} - 2]^{-\frac{1}{1+\lambda}}$, and hence $C(V_1, V_2, V_3) = U_1 U_2^{1/(1+\lambda)} U_3^{1/(1+\lambda)}$ is distributed as $K_3(p) = p[1 + \frac{(1-p)^{-\lambda}}{\lambda} + \frac{(1+\lambda)(1-p^{-\lambda})}{(1-p^{-\lambda})}]$ (see Barbe et al. [1996]).
4. Maximum likelihood estimation

We consider the three families (2), (3), and (5). Within the range of possible values of the dependence parameter, a random sample of size 500 of couples of independent uniform(0,1) is generated and transformed, according to proposition 4, to get independent random samples for each family of copulas.

(1) For the family (2), the likelihood function is

\[ L = (1 - \lambda)^{n_0} \lambda^{n - n_0}, \]

where \( n_0 = \# \{ i : P_i = 0 \} \). The maximum likelihood estimate is \( \hat{\lambda} = 1 - n_0/n \), and \( n - n_0 \) is distributed as \( Bin(n, \lambda) \). Therefore we have \( E(\lambda) = \lambda \) with variance \( \frac{\lambda(1-\lambda)}{n} \). (See figure 1). The 95% asymptotic confidence intervals for the maximum likelihood estimates of the association parameter from (2) exhibit a good coverage in the range for the association parameter.

(2) For the family (3), the likelihood function based on (9) is easily maximized numerically. The 95% asymptotic confidence intervals from (3) of the ml estimates clearly underestimate the true value of the association parameter. Moreover the asymptotic variance tend to increase with the association parameter. (See figure 2)
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Figure 3. ml confidence intervals for (5)

(3) For the family of copulas (5), the likelihood function is

\[ L = n \log(1 + 1/\lambda) + \sum_{i=1}^{n} \log(1 - \mu_i^\lambda). \]

The likelihood equation is \( \frac{n}{\lambda(1 + \lambda)} + \sum \frac{\mu_i^\lambda \log(p_i)}{1 - \mu_i^\lambda} = 0 \) which has to be solved numerically. The 95% asymptotic confidence interval presents a good coverage for the association parameter within a reasonable range for association parameter, but the asymptotic variance based is fairly large for small values for the association parameter. (See figure 3)

In conclusion maximum likelihood estimation based on the pit is more reliable for singular families of distributions.

References


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