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Modelling of convective phenomena in forest fire

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Abstract. We present a model coupling the fire propagation equations in a bidimensional domain representing the surface, and the air movement equations in a three dimensional domain representing an air layer. As the air layer thickness is small compared with its length, an asymptotic analysis gives a three dimensional convective model governed by a bidimensional equation verified by a stream function. We also present the numerical simulations of these equations.

Modelización de la convección en incendios forestales

Resumen. Presentamos un modelo que acopla las ecuaciones de propagación de incendios en un dominio bidimensional que representa la superficie y las ecuaciones de movimiento del aire en un dominio tridimensional que representan una capa de aire. Como el grosor de la capa de aire es pequeño comparado con su longitud, un análisis asintótico da un modelo convectivo tridimensional gobernado por una ecuación bidimensional sobre una función de corriente. Además se presentan simulaciones numéricas de estas ecuaciones.

1. Introduction

We present a model coupling the fire propagation equations in a bidimensional domain representing the surface, and the air movement equations in a three dimensional domain representing an air layer. As the air layer thickness is small compared with its length, an asymptotic analysis gives a three dimensional convective model governed by a bidimensional equation verified by a stream function k. We also present the numerical simulations of these equations.

More precisely, we compute explicitly the three dimensional air velocity as a function of the vertical coordinate z, the stream function $k(t, x)$, the surface temperature $q(t, x)$ and the surface height $h(x)$, where $x = (x_1, x_2)$ are the horizontal coordinates.

The combustion model presented here is based on the model proposed in Asensio and Ferragut [1] incorporating solid phase and gaseous phase. The convection model is an adaptation to the forest fires convective phenomena of the shallow water models proposed in Bresch et al. [3].

2. Combustion equations

Lets $d \subset \mathbb{R}^2$ be a bidimensional bounded domain, representing the projection of the three dimensional geographical surface, x be any of its points and t be the time. We use small letters for quantities concerning the bidimensional problem, and capital letters for the tridimensional problem.

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Fire development is described by the amount of solid fuel, gaseous fuel and oxygen, and also by the temperature. We now give the corresponding nondimensional equations.

The solid fuel amount y_s is driven by

$$\partial_t y_s = -\beta_s y_s e^{-\frac{\gamma_s}{q}} \tag{1}$$

where q is the temperature and $\partial_t = \partial/\partial_t$. The right-hand side represents the solid fuel transform into gaseous fuel due to the pyrolysis.

The gaseous fuel amount y_g is driven by

$$\partial_t y_g + v \cdot \nabla_x y_g - \kappa_g \Delta_x y_g = -\alpha_g y_g y_o e^{-\frac{\gamma_g}{q}} + \beta_s y_s e^{-\frac{\gamma_s}{q}} \tag{2}$$

where v is the wind. The first term of the the right-hand side represents the amount of gaseous fuel burned with the oxygen. The oxygen amount y_o is driven by

$$\partial_t y_o + v \cdot \nabla_x y_o - \kappa_o \Delta_x y_o = -\alpha_o y_g y_o e^{-\frac{\gamma_g}{q}}. \tag{3}$$

Finally, the temperature q is driven by the equation

$$\partial_t q + v \cdot \nabla_x q - \nabla_x \cdot (\kappa_q q^3 \nabla_x q) = \mu y_g y_o e^{-\frac{\gamma_g}{q}}, \tag{4}$$

where radiation is modelled as in Weber [7].

We complete these equations with the following initial conditions

$$y_s|_{t=0} = y_s^0, \quad y_g|_{t=0} = 0, \quad y_o|_{t=0} = \chi, \quad q|_{t=0} = q^0, \tag{5}$$

where y_s^0 is the initial solid fuel amount, q^0 the initial temperature, that we suppose known, and χ is the initial oxygen amount, that is constant; and with the boundary conditions

$$y_g|_{\partial d} = 0, \quad y_o|_{\partial d} = \chi, \quad q|_{\partial d} = q^0|_{\partial d} \tag{6}$$

Greek letters represent several strictly positives coefficients, non depending on t or x.

Remark 1 There is no boundary condition on y_s because (1) is an ordinary differential equation in t, for fixed x and q. ■

3. Convection equations

Lets consider now the three dimensional domain

$$D = \{(x, z) : x \in d,\ h(x) < z < \delta\}$$

representing the air layer. We assume that the ratio of the height to the width, δ, is very small, and the height of the surface at point x, $h(x)$, is smaller than δ. We denote by an index $_{xz}$ the three dimensional operators, that is, $\nabla_{xz} = (\partial_{x_1}, \partial_{x_2}, \partial_z)$ and $\Delta_{xz} = \partial^2_{x_1x_1} + \partial^2_{x_2x_2} + \partial^2_{zz}$.

The air velocity $U = (U_1, U_2, U_3)$ and the potential P satisfy the Navier–Stokes equations. On one hand, the momentum equation, reads

$$\partial_t U + U \cdot \nabla_{xz} U - \frac{1}{\mathrm{Re}} \Delta_{xz} U + \nabla_{xz} P = \varphi Q e_3 \tag{7}$$

where Q is the temperature and $e_3 = (0, 0, 1)$. The right-hand side represents the Archimedes force due to the expansion under the effect of heat, which has the form $c(Q - Q_0)$. Notice that the part of the force corresponding to the reference temperature Q_0 is $\varphi Q_0 e_3 = \nabla_{xz}(\varphi Q_0 z)$, that allows to include it into the potential term $\nabla_{xz} P$. The density variations due to the temperature have been neglected into the other terms of the equation. On the other hand, the air compressibility is also neglected, so that

$$\nabla_{xz} \cdot U = 0. \tag{8}$$

In order to specify the boundary conditions, we decompose the boundary into $\partial D = S \cup A \cup L$, where the

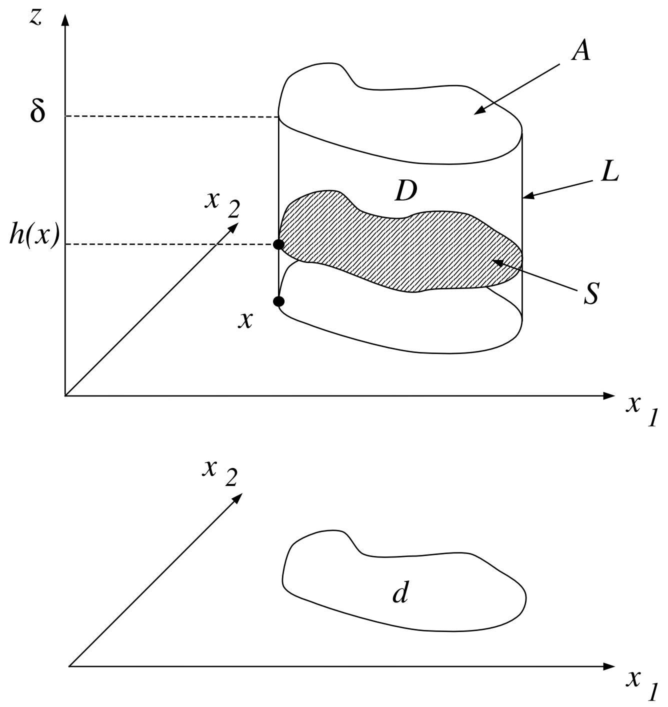

Figure 1. Domains in 2D and 3D.

surface S, the air upper boundary A and the air side boundary L are respectively

$$S = \{(x,z) : x \in d,\ z = h(x)\}, \quad A = \{(x,z) : x \in d,\ z = \delta\},$$
$$L = \{(x,z) : x \in \partial d,\ h(x) < z < \delta\}.$$

On surface S,

$$U \cdot N = 0, \quad \frac{\partial U}{\partial N}\Big|_{\text{tang}} = \zeta U, \tag{9}$$

where N is the inner unit normal vector field to ∂D, and the subscript $_{\text{tang}}$ denotes the tangential component, that is, $f_{\text{tang}} = f - (f \cdot N)\, N$ for any vector field f. The first condition in (9) express that the air does not cross the surface, and the second condition express that the traction force is proportional to the tangential velocity. Indeed, the tangential stress $\sigma N|_{\text{tang}}$ is equal to $c\frac{\partial U}{\partial N}|_{\text{tang}}$ when $U \cdot N = 0$.

On the air upper boundary A,

$$U \cdot N = 0, \quad \frac{\partial U}{\partial N}\Big|_{\text{tang}} = 0, \tag{10}$$

that is, the air does not cross this boundary and there is no traction forces. Since this part of the boundary is horizontal, this condition reads $U_3 = 0$ and $\nabla_z U_1 = \nabla_z U_2 = 0$.

On the air side boundary L,

$$U\big|_L = (v_m, 0) \tag{11}$$

more precisely $U(t,x,z) = (v_m(t,x), 0)$ where v_m is the meteorological wind, that we assumed to be known, horizontal, non depending on z and with null flux, that is,

$$\partial_z v_m = 0, \qquad \int_{\partial d} (\delta - h)\, v_m \cdot n \, ds = 0 \tag{12}$$

where $n = (n_1, n_2)$ is the inner unit normal vector field to ∂d.

We complete these equations with the initial condition

$$U_{|t=0} = U_0 \tag{13}$$

where U_0 is the initial velocity, that we assume to be known.

Remark 2 The equations (7) to (13) are well posed: for given Q, v_m and U_0, there exists a solution (U, P) if the data are "smooth enough", that is, if the data are given in appropriate functional spaces; the solution is unique (up to an additive constant for P) if the data are "small enough ". This can be proved as in Bresch et al., [3], where a similar problem is considered. ■

Remark 3 We prove in § 4. that hypothesis (12) is needed when the condition on L is given by (11). Nevertheless, it is not necessary that the meteorological wind be neither horizontal, nor independent on z ; that is, we replace (11) by $U|_L = u_m$ with u_m such that $\int_L u_m \cdot N\, dS = 0$.

But, in the asymptotic convection model of § 5. the condition on L will only concern the normal component of the vertical average of the horizontal velocity, *cf.* (24) (because equation (7) degenerates, on vertical direction becomes (19) and (20)). The vertical component and vertical variations on L of the meteorological wind, have not influence on the model to be solved, and therefore they can be chosen nulls. ■

4. Horizontal flux

An important property of such a velocity field is that its horizontal flux is incompressible. More precisely, we distinguish the vertical velocity from the horizontal one denoting

$$V = (U_1, U_2), \quad W = U_3$$

and we define the horizontal flux on a point $x \in d$ at time t by

$$\overline{V}(t,x) = \int_{h(x)}^{\delta} V(t,x,z)\, dz. \tag{14}$$

The incompressibility and the fact that the air does not cross S and L involve

$$\nabla_x \cdot \overline{V} = 0 \tag{15}$$

that is, this flux is incompressible. Indeed, (8) reads $\nabla_x \cdot V + \partial_z W = 0$, and thus

$$\begin{aligned}
\nabla_x \cdot \int_{h(x)}^{\delta} V(t,x,z)\, dz &= \int_{h(x)}^{\delta} \nabla_x \cdot V(t,x,z)\, dz - V(t,x,h(x)) \cdot \nabla_x h(x) \\
&= -W(t,x,\delta) + W(t,x,h(x)) - V(t,x,h(x)) \cdot \nabla_x h(x) \\
&= (U \cdot N)(t,x,\delta) + c(x)(U \cdot N)(t,x,h(x))
\end{aligned}$$

that is null due to (10), that shows (15).

Remark 4 Lets verify that the hypothesis (12) is necessary for the equations (8) to (11) have a solution. By the Stokes formula and (15), $\int_{\partial d} \overline{V} \cdot n \, ds = \int_d \nabla_x \cdot \overline{V} \, dx = 0$. Then it follows (12) as $\overline{V} = (\delta - h)\, v_m$ on ∂d due to (11). ■

Remark 5 As the meteorological wind is almost constant on a fire scale (it is an expected wind, not a measured wind) we can take it in general in the form

$$v_m(t, x) = \frac{c_D m(t)}{\delta - h(x)} \tag{16}$$

where m is the "global" expected wind in the area and c_D is the average value of $1/(\delta - h)$ on ∂d, that is $c_D = \int_{\partial d} 1/(\delta - h)\, ds \,/\, \int_{\partial d} ds$, and so $c_D/(\delta - h)$ has an average 1. Then v_m is the "local" corresponding wind, adapted to the surface ; Moreover, it verifies the hypothesis (12) as $\int_{\partial d} n \, ds = 0$.

Particularly, the hypothesis (12) is satisfied for all meteorological constant wind v_m when the surface is flat, that is when $h = 0$, and then $c_D = 1$ and $m = v_m$. ■

5. Convection asymptotic model

We now use the fact that the layer thickness where the convective air motion due to the fire takes place , is small in relation to its width, that is

$$\delta \ll 1. \tag{17}$$

We also assume that the wind in not too strong, more precisely

$$\delta^2 \mathrm{Re} \ll 1. \tag{18}$$

Preserving only the dominant terms and rescaling P, then equations (7) and (8) give, as we see on next section,

$$-\partial^2_{zz} V + \nabla_x P = 0, \tag{19}$$

$$\partial_z P = \lambda Q, \tag{20}$$

$$\nabla_x \cdot V + \partial_z W = 0. \tag{21}$$

where $\lambda = \varphi \mathrm{Re}$. The conditions (9), (10) and (11) give, particularly,

$$\partial_z V = \zeta V, \quad (V, W) \cdot N = 0 \qquad \text{on } S, \tag{22}$$

$$\partial_z V = 0, \quad W = 0 \qquad \text{on } A, \tag{23}$$

$$\overline{V} \cdot n = (\delta - h) v_m \cdot n \qquad \text{on } \partial d. \tag{24}$$

Remark 6 Equations (19) to (24) are well posed : for given Q and v_m, there exists a unique solution (V, W, P) (except for P which is unique up to an additive constant) that we compute on sections § 8. and 9. ■

6. Justification of the asymptotic analysis

Obtaining the incompressibility equation (21)

Notice that $x \sim 1$ (as the characteristic length is the one of d) and $z \sim \delta$, so

$$\nabla_x \sim 1, \quad \partial_z \sim 1/\delta. \tag{25}$$

For an appropriate choice of the characteristic velocity $(V, W) \sim 1$, so the incompressibility condition (8), which reads $\nabla_x \cdot V + \partial_z W = 0$, gives

$$V \sim 1, \quad W \sim \delta, \tag{26}$$

and (21) holds.

Obtaining the vertical diffusion equation (19)

We now consider the operators appearing in the momentum equation (7). First $t \sim 1$ (for an appropriate choice of the characteristic time), so $\partial_t \sim 1$. Second, with (25) and (26), $U \cdot \nabla_{xz} = V \cdot \nabla_x + W\partial_z \sim 1$. Moreover, (25) implies, $\Delta_{xz} = \Delta_x + \partial^2_{zz} \sim \partial^2_{zz}$ so with (18), $\Delta_{xz}/\mathrm{Re} \sim 1/(\delta^2 \mathrm{Re}) \gg 1$. All this gives

$$\partial_t + U \cdot \nabla_{xz} - \frac{1}{\mathrm{Re}}\Delta_{xz} \sim -\frac{1}{\mathrm{Re}}\partial^2_{zz} \tag{27}$$

As the horizontal component of Qe_3 is null, then the horizontal component of (7), in the limit, is

$$-\frac{1}{\mathrm{Re}}\partial^2_{zz}V + \nabla_x P = 0. \tag{28}$$

Here, the term $\nabla_x P$ cannot be neglected because on the contrary we would have $\partial^2_{zz}V = 0$, and V would be determined by the conditions on S and A, so it would not depend on either meteorological wind or temperature, and this is absurd. The term $\frac{1}{\mathrm{Re}}\partial^2_{zz}V$ cannot be neglected either because on the contrary, V would be solely governed by the incompressibility equation ant it would be partly indeterminate. As no term of (28) is negligible, this equation can be written as (19) using $\mathrm{Re}P$ a the new potential.

Obtaining the hydrostatic equilibrium equation (20)

From (27), and the above rescaling of P, the vertical component of (7) in the limit gives

$$-\partial^2_{zz}W + \partial_z P = \lambda Q. \tag{29}$$

From (25) and (26), $\partial^2_{zz}W \sim 1/\delta$. From (19) $P \sim 1/\delta^2$ and then $\partial_z P \sim 1/\delta^3$. As this term control the previous one, (29) reduces to $\partial_z P = \lambda Q$, that is (20).

Obtaining the conditions on the surface (22)

The slide condition in (9) reads $U \cdot N = V \cdot N_x + WN_z = 0$ where $N_x = -\nabla_x h/C$, $N_z = -1/C$ and $C = (1 + |\nabla_x h|^2)^{-1/2}$. As $h \sim \delta$, we have

$$N_x \sim \delta, \quad N_z \sim 1.$$

From (25) and (26), it follows that $V \cdot N_x \sim WN_z$, so the slide condition is preserved. It can be written $(V, W) \cdot N = 0$.

Lets consider the traction condition in (9), where

$$\frac{\partial U}{\partial N}\Big|_{\mathrm{tang}} = \Big(\frac{\partial V}{\partial N} - \Big(\frac{\partial U}{\partial N} \cdot N\Big) N_x, \frac{\partial W}{\partial N} - \Big(\frac{\partial U}{\partial N} \cdot N\Big) N_z\Big).$$

As $\partial/\partial N = N \cdot \nabla_{xz} = N_x \cdot \nabla_x + N_z\partial_z \sim \partial_z$ then $\partial U/\partial N \cdot N \sim \partial_z U \cdot N = \partial_z V \cdot N_x + \partial_z W N_z \sim 1$ and then $\partial U/\partial N|_{\mathrm{tang}} \sim (\partial_z V, 0)$. Moreover $U \sim (V, 0)$, so the traction condition $\partial U/\partial N|_{\mathrm{tang}} = \zeta U$ gives to the limit $\partial_z V = \zeta V$.

Obtaining the equilibrium conditions on the air (23)

It is exactly condition (10) since A is an horizontal plane.

Obtaining the lateral flux condition (24)

The equality with the meteorological wind (11) particularly shows that $V = v_m$ on L. Integrating over z between $h(x)$ and δ, for a fixed $x \in \partial d$, it follows $\overline{V} = (\delta - h)\, v_m$, and so $\overline{V} \cdot n = (\delta - h)\, v_m \cdot n$, that is (24).

We cannot impose any other conditions on L because, for fixed Q and x, equations (19) and (20) determine $V(x, z)$ up to a polynomial function $az^2 + bz + c$, and conditions (22), (23) and (24) determine a, b and c. Likewise, (21) and (23) determine $W(x, z)$.

The approximation of the solution of the convective equations (7) – (13) by the solution of (19) – (24) is only possible up to a limit layer on L. This limit layer has no physical meaning. As an evidence notice that given a domain $d' \supset d$ such that $\partial d' \neq \partial d$, there is no limit layer on d : it has been "displaced" on d'. The limit layer appears because, on a thin air layer, the vertical wind variations are asymptotically determined by the conditions "on the upper and lower boundaries". In more mathematic terms, the operator degenerates into the vertical direction, and this involves a loss of boundary conditions on the vertical boundary. The limit operator does not contain the horizontal diffusion term that allowed to fit the initial problem solution to any given boundary condition.

Remark 7 The previous asymptotic analysis can be mathematically justified when $\delta \mathrm{Re} \ll 1$, which is a more restrictive condition that (18). The idea of the proof is to use a vertical scaling with rate δ in order to get a problem in a fixed domain independent of δ with coefficients proportional to some powers of δ, for which convergence follows from some energy estimations.

This study is made in Bresch et al. [3] for a similar model to the one described here. In fact, the problem described there is a model of a lake, where the depth ($\delta - h$ in the present model) is null on ∂d (the lake shore). This involves to use weighted spaces, so the demonstration is more complex than in the present case. On the contrary, there is no limit layer, since depth is null on ∂d, so the demonstration is more simple than in the present case.

This is also solved in Bayada et al. [2] for a similar model of lubrication. ■

7. Coupling with temperature

The temperature q considered in the combustion equations (1) to (3) is obviously the value of Q on the surface, that is $q(t, x) = Q(t, x, h(x))$. We assume that

$$Q(t, x, z) = q(t, x)\frac{\delta - z}{\delta - h(x)} \tag{30}$$

that is, the air temperature linearly decreases with the height and vanishes on the upper boundary of the air layer.

The velocity v considered in the combustion equations is the value of V on the surface, that is the horizontal component of the wind velocity U on the surface; in other words,

$$v(t, x) = V(t, x, h(x)).$$

Indeed d is the projection of the surface S (that is a $\mathbb{R}^3$ surface) on the x-plane, so the propagation velocity v in d, at a point x at time t, is the projection on S of propagation velocity U, that is V, at point $(x, h(x))$ and at time t.

Remark 8 The hypothesis (30) about the linear variation of the temperature as a function of height z could be justified by an asymptotic analysis of heat diffusion equations. ■

8. Solution of the convection asymptotic model

Lets compute explicitly $P(t,x,z)$ and $V(t,x,z)$ in terms of z, $q(t,x)$, $h(x)$ and $\nabla_x p(t,x)$ where p is a 2D potential.

For a fixed x, equation (20), that is $\partial_z P = \lambda Q$, with Q given by (30) provides

$$P(t,x,z) = p(t,x) + \frac{\lambda q(t,x)}{\delta - h(x)}\big(\delta z - \tfrac{1}{2}z^2\big) \tag{31}$$

for some $p(t,x)$.

Equation (19), that is $-\partial^2_{zz} V = \nabla_x P$, together with conditions $\partial_z V(t,x,\delta) = 0$ and $\partial_z V(t,x,h(x)) = \zeta V(t,x,h(x))$ included in (23) and (22), then provides

$$\begin{aligned} V(t,x,z) = {} & \big(\tfrac{1}{2}z^2 - \delta z - \tfrac{1}{2}h^2(x) + (\delta+\xi)h(x) - \xi\delta\big)\nabla_x p(t,x) \\ & + \big(-\tfrac{1}{24}z^4 + \tfrac{1}{6}\delta z^3 - \tfrac{1}{3}\delta^3 z + \tfrac{1}{24}h^4(x) \\ & - \tfrac{1}{6}h^3(x)(\delta+\xi) + \tfrac{1}{2}\xi\delta h^2(x) + \tfrac{1}{3}\delta^3 h(x) - \tfrac{1}{3}\xi\delta^3\big)\nabla_x \widehat{q}(t,x) \end{aligned} \tag{32}$$

where

$$\xi = \frac{1}{\zeta}, \quad \widehat{q}(t,x) = \frac{\lambda q(t,x)}{\delta - h(x)}. \tag{33}$$

Notice that $\widehat{q}(t,x) = (\lambda/\delta)\,Q(t,x,0)$, *cf.* (30). Up to the factor λ/δ, it is the temperature at a height $z = 0$.

9. Stream function 2D equations

We now assume that

$$\partial d \text{ is connected.} \tag{34}$$

Since it is bounded, d is then connected, simply connected (this means that it does not possess "holes" because dimension is 2) and it is "inside" its boundary ∂d. The hypothesis (12), that is $\int_{\partial d}(\delta - h)\, v_m \cdot n\, ds = 0$ gives the existence of a function v_m^* such that

$$\nabla_x \cdot v_m^* = 0, \quad v_m^* \cdot n = (\delta - h)\, v_m \cdot n \text{ on } \partial d. \tag{35}$$

Since $\nabla_x \cdot \overline{V} = 0$, *cf.* (15), it follows that $\nabla_x \cdot (\overline{V} - v_m^*) = 0$ therefore, again thanks to (34), *cf.* for example Simon, [6], there exists a function k such that

$$\nabla_x^{\perp} k = \overline{V} - v_m^*, \quad k|_{\partial d} = 0, \tag{36}$$

where the superscript $^{\perp}$ denotes a rotation of $\pi/2$, that is $(f_1, f_2)^{\perp} = (-f_2, f_1)$ for any vector f. Integrating (32) with respect to z, from $h(x)$ to δ, we obtain, *cf.* definition (14),

$$\overline{V} = -a\nabla_x p - b\nabla_x \widehat{q} \tag{37}$$

where

$$\begin{aligned} a(x) &= \tfrac{1}{3}\delta^3 + \xi\delta^2 - \delta(\delta + 2\xi)h(x) + (\delta+\xi)h^2(x) - \tfrac{1}{3}h^3(x) \\ &= \tfrac{1}{3}(\delta - h(x))^2(3\xi + \delta - h(x)), \end{aligned} \tag{38}$$

$$\begin{aligned} b(x) = {} & \tfrac{2}{15}\delta^5 + \tfrac{1}{3}\xi\delta^4 - \tfrac{1}{3}\delta^3(\delta+\xi)h(x) + \tfrac{1}{6}\delta^2(\delta - 3\xi)h^2(x) \\ & + \tfrac{1}{6}\delta(\delta + 4\xi)h^3(x) - \tfrac{1}{6}(\delta+\xi)h^4(x) + \tfrac{1}{30}h^5(x) \\ = {} & \tfrac{1}{30}(\delta - h(x))^2(2\delta^2(2\delta + 5\xi) - 2\delta(\delta - 5\xi)h(x) - (3\delta + 5\xi)h^2(x) + h^3(x)). \end{aligned} \tag{39}$$

This implies

$$-\nabla_x \cdot \Big(\frac{1}{a}\nabla_x k\Big) = f, \quad k|_{\partial d} = 0, \tag{40}$$

where

$$f = \nabla_x^\perp \cdot \frac{b\nabla_x \widehat{q} + v_m^*}{a}. \tag{41}$$

Indeed, (36) and (37) give

$$\nabla_x^\perp k = -(a\nabla_x p + b\nabla_x \widehat{q} + v_m^*) \tag{42}$$

therefore, dividing by a, taking the orthogonal $^\perp$, and taking into account that $f^{\perp\perp} = -f$, we get $-\nabla_x k/a = -(\nabla_x p + (b\nabla_x \widehat{q} + v_m^*)/a)^\perp$. Taking the divergence, this gives the first equation of (40), with $f = -\nabla_x \cdot (\nabla_x p + (b\nabla_x \widehat{q} + v_m^*)/a)^\perp$. It yields (41) because $\nabla_x \cdot \nabla_x^\perp = 0$ and $-\nabla_x \cdot g^\perp = \nabla_x^\perp \cdot g$. The second equation of (40) follows from (36).

Remark 9 Equation (40) is elliptic because, by (38),

$$a(x) \geq \eta > 0$$

where $\eta = \xi\underline{\delta}^2 + \frac{1}{3}\underline{\delta}^3$ and $\underline{\delta} = \delta - \sup_{x\in d} h(x) > 0$. Therefore it has a unique solution k. ∎

10. Velocity on the surface

The velocity on the surface is $v(t,x) = V(t,x,h(x))$, therefore (32) gives

$$v(t,x) = -\xi\,(\delta - h(x))\,\big(\nabla_x p(t,x) + \big(\tfrac{1}{3}\delta^2 + \tfrac{1}{3}\delta h(x) - \tfrac{1}{6}h^2(x)\big)\nabla_x \widehat{q}\big)$$

and with (42) it follows

$$v(t,x) = \frac{\xi}{c(x)}(\nabla_x^\perp k(t,x) + e(x)\nabla_x \widehat{q}(t,x) + v_m^*(t,x)) \tag{43}$$

where

$$c(x) = \tfrac{1}{3}(\delta - h(x))\,(\delta + 3\xi - h(x)), \quad e(x) = \tfrac{1}{45}(\delta - h(x))^5. \tag{44}$$

Remark 10 For a given meteorological wind v_m on ∂d, verifying $\int_{\partial d}(\delta - h)\, v_m \cdot n\, ds = 0$, there exist infinitely many solutions v_m^* of (35) however all of them give the same value to v.

Indeed, the general solution of (35) is $v_m^* = v_0 + \nabla^\perp g$ where v_0 is a particular solution of (35) and where g is any function null on ∂d. Then (40)–(41) can be written

$$-\nabla_x \cdot \Big(\frac{1}{a}\nabla_x (k+g)\Big) = \nabla_x^\perp \cdot \frac{b\nabla_x \widehat{q} + v_0}{a}, \quad (k+g)|_{\partial d} = 0,$$

and then $k+g$ does not depend on the choice of g. In other words, all v_m^* give the same value to $k+g$, and also to $\nabla_x^\perp(k+g) + v_0$, that is to $\nabla_x^\perp k + v_m^*$, and also to $\nabla_x p$ due to (42), and also to the velocity V thank to (32), and particularly of the flux $\overline{V}$ and of the velocity on the surface v, *cf.* (43).

The same conclusion could be obtained observing that conditions $\nabla_x \cdot \overline{V} = 0$ and $\overline{V} \cdot n = (\delta - h)v_m \cdot n$ on ∂d give, with (37),

$$-\nabla_x \cdot (a\nabla_x p) = \nabla_x \cdot (b\nabla_x \widehat{q}) = 0, \quad \frac{\partial p}{\partial n} = -\frac{b}{a}\frac{\partial \widehat{q}}{\partial n} + \frac{\delta - h}{a} v_m \cdot n$$

that determine $\nabla_x p$ in a unique way, that is, independently of the choice of v_m^*. ∎

11. Resume of the convection model

For a fixed time t, to any temperature field q defined on d, we associate a velocity field v defined on d by

$$v = \frac{\xi}{c}(\nabla_x^{\perp} k + e\nabla_x \widehat{q} + v_m^*)$$

where k is the unique solution of

$$-\nabla_x \cdot \left(\frac{1}{a}\nabla_x k\right) = \nabla_x^{\perp} \cdot \frac{b\nabla_x \widehat{q} + v_m^*}{a}, \quad k_{|\partial d} = 0,$$

and where $\widehat{q} = q\lambda/(\delta - h)$ is the normalized temperature. Functions $a(x)$, $b(x)$, $c(x)$ and $e(x)$, that model the influence of the surface, are defined by (38), (39) and (44).

The meteorological wind v_m is a data on ∂d such that $\int_{\partial d}(\delta - h)\, v_m \cdot n\, ds = 0$, and v_m^* is any velocity field defined in d such that

$$\nabla_x \cdot v_m^* = 0, \quad v_m^* \cdot n = (\delta - h)\, v_m \cdot n \text{ on } \partial d.$$

This gives

$$v = \mathcal{L}(q, v_m, h)$$

where $\mathcal{L}$ is a time t independent operator, affine with respect to q and v_m, and non linear with respect to h.

Remark 11 It can be shown that, given h in $\mathcal{C}(d)$, the application $(q, v_m) \mapsto v$ is continuous from $H^1(d) \times (L^2(\partial d))^2$ into $(L^2(d)^2$. If the boundary conditions are defined "in the sense of H_0^1", it is enough that d be an open bounded set. ■

12. About the validity of the asymptotic model

Condition (18), that is $\delta^2 \mathrm{Re} \ll 1$, makes negligible the nonlinear terms as seen in § 6. This is the Reynolds approximation, classic in lubrification, see Bayada et al. [2].

If Reynolds number is not small enough for that, it can be lowered as it is made in oceanography when the molecular viscosity is replaced by the turbulent viscosity. This is equivalent to replace the nonlinear transport terms by linear diffusion terms. Briefly, we can say that, in oceanography, the vortex of size lower than mesh size are removed, whereas here, are removed the vortex of size lower than the height of the air layer perturbed by the fire. In oceanography, this is justified by the insufficient of the actual computer; here, the aim is to obtain a velocity field explicitly computable in terms of z. In both cases, a part of the effects due to the turbulence (that can be interpreted as instabilities or multiple bifurcations due to the nonlinearity) are lost.

The obtained approximation is reasonable because it takes into account, on one hand, Archimedes force due to the air expansion under the effect of heat, and on the other hand, the local movement transmission generated in the fluid set under the effect of the incompressibility. These two phenomena roughly determine the air movement.

13. Numerical method

The numerical method employed here is an adaptive finite element method combined with a time-stepping splitting technique. Each time step is subdivided in three sub-steps, correspondingly equations (1)–(4) are split into a radiation equation, a convection equation and a diffusion equation. The radiation step is solved by an implicit Euler method, the convection step by a characteristic method and the diffusion step using P1-Lagrange finite element approximation. Previously, the velocity at the surface has been computed solving the stream function problem (40) and using (43). At each step, a mesh adaptation is performed. This numerical algorithm has been implemented using freeFEM+, a finite element package of O. Pironneau and F. Hecht [5].

14. Numerical simulation

As an application, we present the numerical results of a simulated combustion in a normalized rectangle of size 2×1, for an initial fire focus and uniform wind given on the boundary. There is a hill in the center of the rectangle.

The nondimensional activation energy for gas combustion is $\gamma_g = 33$ and for solid pyrolysis $\gamma_s = 20$. We also have taken for diffusion $\kappa_g = \kappa_o = 0.1$ and $\kappa_q = 0.1$ for heat radiation coefficient in (4). The other parameters of the combustion equations has been taken equal to unit.

Concerning convection model, the velocity on the boundary is $v_m = (220, 0)$ (which roughly corresponds to a real velocity between $2 - 5$m/s) with a friction coefficient $\zeta = 10$ and $\lambda = 3$ in equation (20). All this values has been chosen in order to have reasonable results.

The initial fire focus is given by the expression

$$q^0(x_1, x_2) = 30e^{-200((x_1-0.25)^2+(x_2-0.5)^2)} \tag{45}$$

After 1000 time steps we obtain the temperature contours shown in figure 2. After this time, a stable fire front has been obtained. Figure 3 shows the contour plot of the x-component of velocity. Observe that the velocity near the fire front and the meteorological wind have opposite direction. Comparing this figure with figure 4, where the fire front is near the top of the hill, we can appreciate the effect of the hill, so that the combined effect of the meteorological wind and the slope of the terrain surface is bigger than the effect of the gradient of the temperature in the fire front. We can appreciate the burned area with the contour lines corresponding to the solid fuel in figure 5. Figures 6 and 7, show the contour lines of the gas fuel and oxygen respectively. In figure 8, a typical adapted mesh is shown.

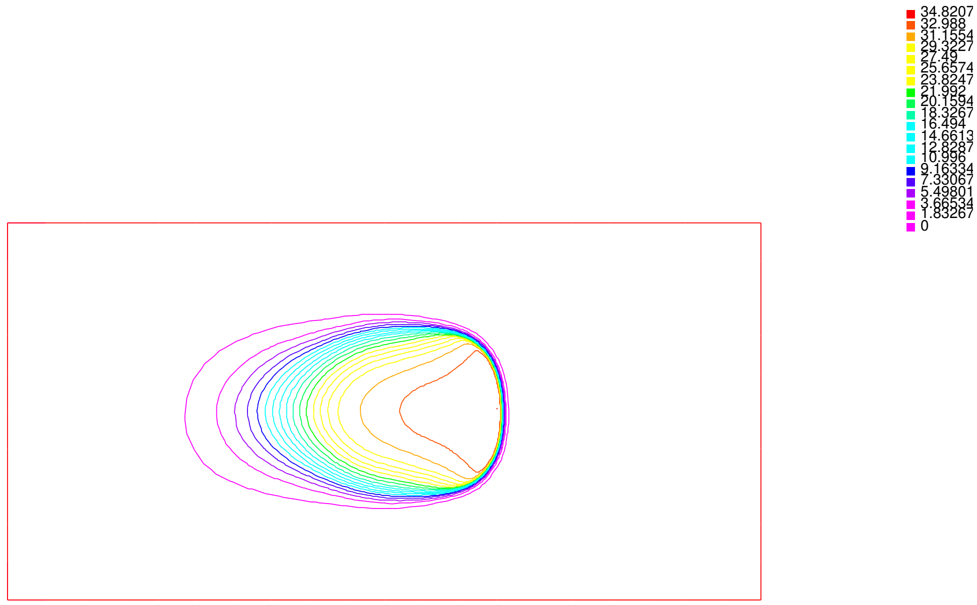

Figure 2. Temperature.

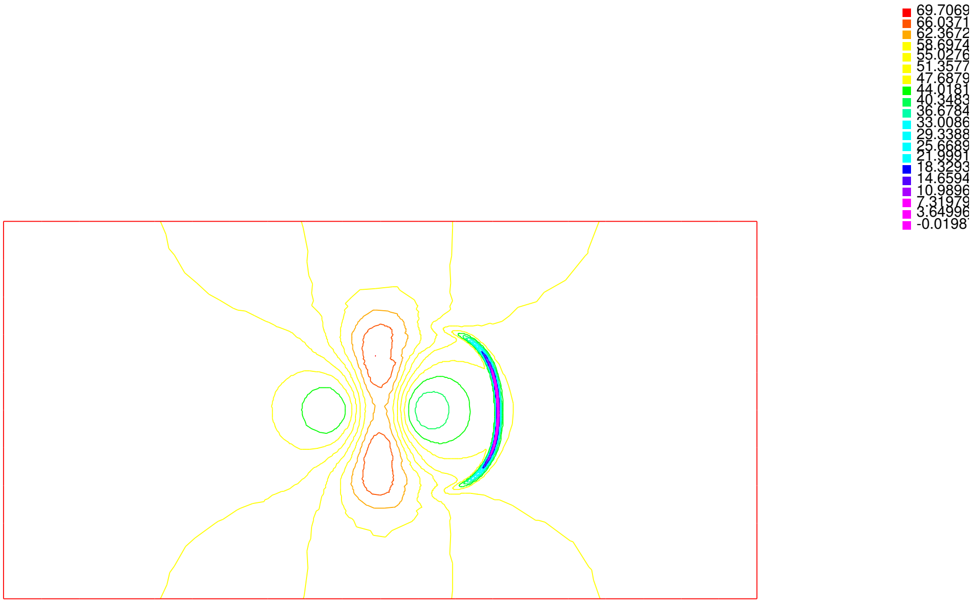

Figure 3. x-component of velocity after 1000 time steps.

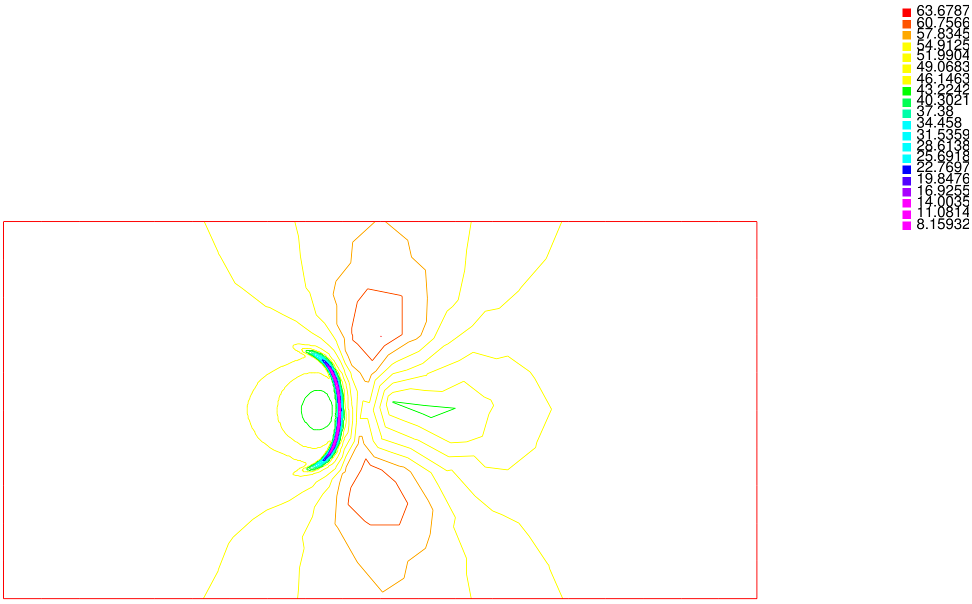

Figure 4. x-component of velocity after 600 time steps.

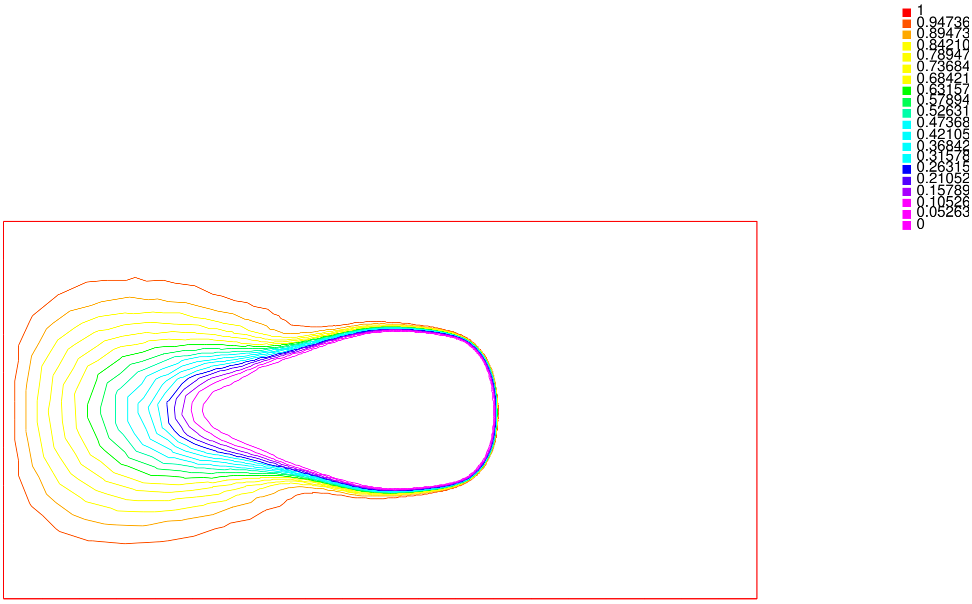

Figure 5. Solid fuel.

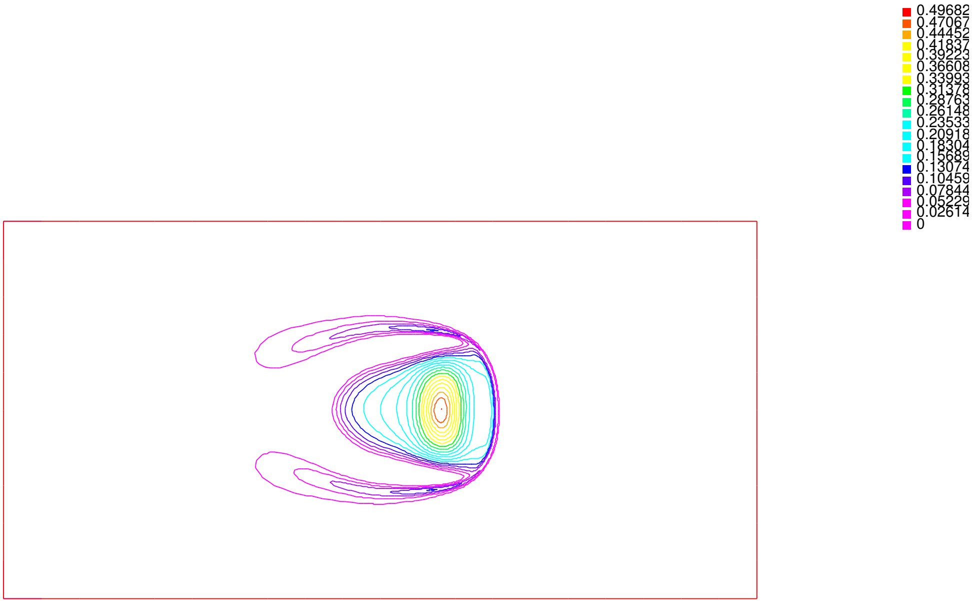

Figure 6. Gaseous fuel.

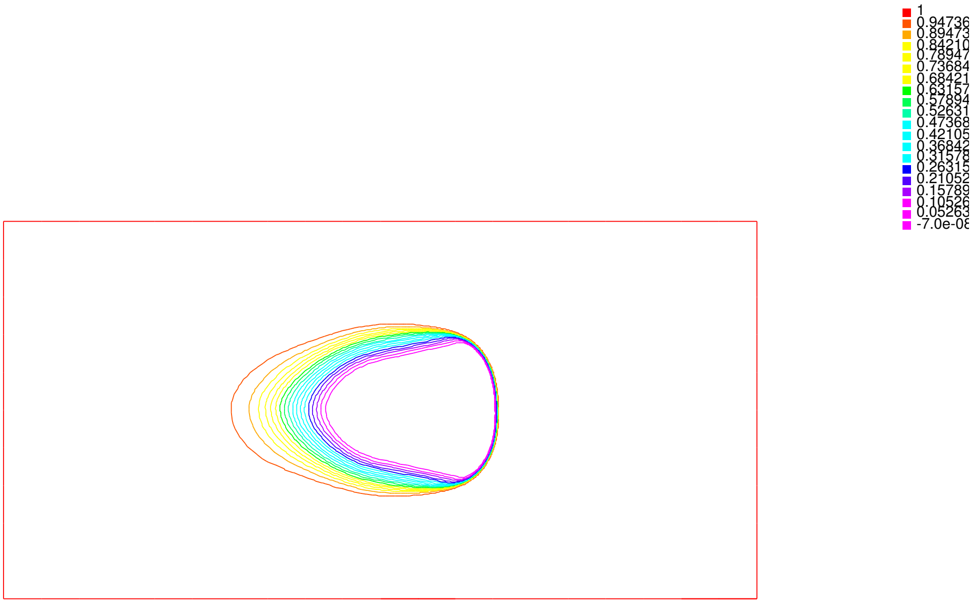

Figure 7. Oxygen.

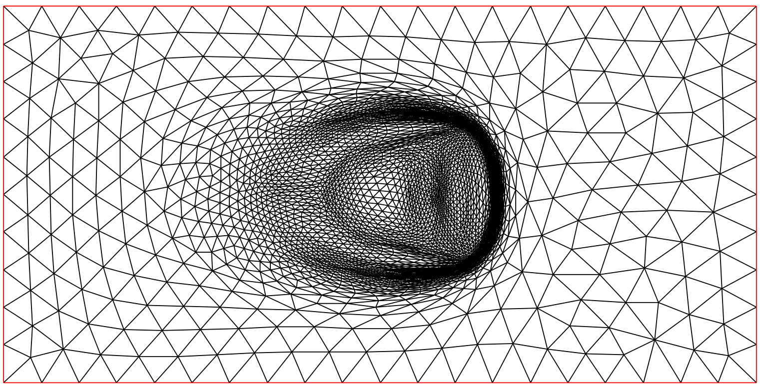

Figure 8. Mesh.

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