RACSAM
Rev. R. Acad. Cien. Serie A. Mat.
VOL. **96** (3), 2002, pp. 401–410
Matemática Aplicada / Applied Mathematics
Artículo panorámico / Survey

A zoology of boundary layers

D. Gérard-Varet and E. Grenier

The authors would like to dedicate this note to Prof. J.-L. Lions, whose lectures at Collège de France motivate and encourage the beginning of this study.

Abstract. In meteorology and maghetohydrodynamics many different boundary layers appear. Some of them are already mathematically well known, like Ekman or Hartmann layers. Others remain unstudied, and can be much more complex. The aim of this paper is to give a simple and unified presentation of the main boundary layers, and to propose a simple method to derive their sizes and equations.

Una zoología de capas finas

Resumen. En meteorología y magnetohidrodinámica aparecen numerosas capas límites. Algunas de ellas son perfectamente conocidas, como es el caso de las capas límites de Ekman y de Hartmann. Otras permanecen desconocidas y pueden ser mucho más complejas. El objetivo de este artículo es ofrecer una presentación unificada de las capas límites más importantes y proponer un método sencillo para obtener sus tamaños y ecuaciones.

1. Introduction

One of the main features in oceanography and meteorology, and also in magneto-hydro-dynamics is the presence of one or more small parameters. Typically in oceanography, after appropriate time and space rescaling, the rotation speed of the Earth (which creates the Coriolis force) is pretty large (10^2 to 10^4), the aspect ratio (ratio between the depth and the horizontal sizes) is small (like a few kilometers over several thousands kilometers), parameters describing the stratification (like the so called Brunt Vassaila frequency) are large. In MHD, in the study of the Earth magnetic field, the rotation speed is even larger (after rescaling, something like 10^8), the strength of the magnetic field is very important (10^8 also).

In the interior of the domain these large parameters lead to some reduced behavior (for instance the velocity field may be independent of some spatial variable), behavior which is often incompatible with boundary conditions. This leads to boundary layers, small zones near the boundary where the system adapts itself to the boundary conditions.

In some cases, the size and equation of the layers are easy to derive, like in the study of viscous perturbations of hyperbolic systems. In other problems, like the one of Ekman layers, we have to find our way in asymptotic expansions, which are more or less well written in the classical physical monographs

Presentado por Haïm Brezis
Recibido: 3 de Julio de 2002. *Aceptado*: 9 de Octubre de 2002.
Palabras clave / Keywords: 76D10, 76N20, 76W05
Mathematics Subject Classifications: Boundary layers, meteorology, magnetohydrodynamics.
© 2002 Real Academia de Ciencias, España.

(see [2, 4]). But many other layers (Stewartson, equatorial layers) are much more subtle. Usually they are studied either by explicit but cumbersome solutions (using Bessel's functions) or by using refined physical arguments, without justification. The fundamental papers of Stewartson [8, 7] and Proudman [5] are typical of such approaches. In a rotating sphere for instance, boundary layers of size $E^{1/2}$, $E^{1/3}$, $E^{1/4}$, $E^{2/5}$, ... are expected, where E is the Ekman number (see figure). Only the $E^{1/2}$ has been fully studied and has a very clear derivation (see for instance [3]).

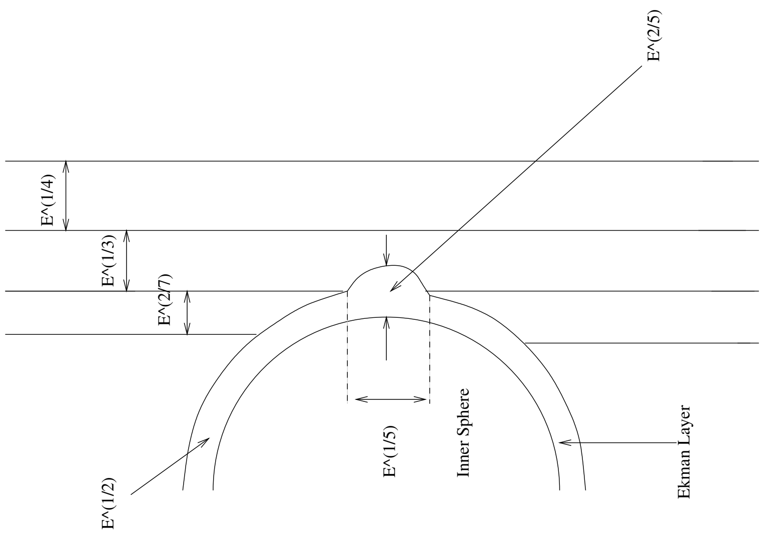

Figure 1. Boundary layers of rotating fluids, near a sphere and at the circumscribing cylinder (following Stewartson [7])

The aim of this paper is to clarify this situation and to give a global method to derive the various layer sizes and equations. We hope this work will help mathematicians to get interested in the numerous existence, unicity and stability related problems. This note will be detailed and parts of it rigorously justified in a forthcoming paper [1].

Let us now introduce two typical systems, on which we will illustrate our method throughout next sections. The first one describes an incompressible viscous fluid in a highly rotating domain. Governing equations are Navier Stokes Coriolis equations

$$\partial_t u + u \cdot \nabla u + \frac{\nabla p}{\varepsilon} - \frac{E}{\varepsilon}\Delta u + \frac{\mathbf{e} \times u}{\varepsilon} = 0, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u = 0 \qquad \text{on} \qquad \partial\Omega, \tag{3}$$

where $\mathbf{e}$ is a given fixed vector $\mathbf{e} = (0, 0, 1)$, E and ε are two small parameters called Ekman and Coriolis numbers. Typically E and ε^2 are of the same order. Domains of interest are $\Omega = \mathbb{T}^2 \times [0, 1]$ (where $\mathbb{T}^2$ is the two dimensional periodic flat torus), $\omega \times [0, 1]$ where ω is a two dimensional domain, $B(0, 1)$ (ball of center 0 and radius 1), $B(0, 1) - B(0, R)$ with $R < 1$ (between two spheres). The main boundary layer is the classical Ekman layer of size $E^{1/2}$. However near vertical boundaries, or near points where the tangent plane of $\partial\Omega$ is vertical, boundary layers of size $E^{1/3}$ and $E^{1/4}$ may appear. More complex layers of size $E^{2/5}$ and $E^{2/7}$ are expected in spherical cases.

The second one is the following MHD system

$$\partial_t u + u \cdot \nabla u + \frac{\nabla p}{\varepsilon} - \Delta u + \frac{\mathbf{e} \times j}{\varepsilon} = 0 \tag{4}$$

$$j = \nabla\phi + u \times \mathbf{e} \tag{5}$$

$$\nabla \cdot u = \nabla \cdot j = 0 \tag{6}$$

$$u = 0, \quad j \cdot n = 0, \quad \text{on } \partial\Omega \tag{7}$$

This system describes an incompressible viscous fluid under the action of a strong magnetic field $\varepsilon^{-1}\mathbf{e} \times j$, where j is the current density, linked to u through Ohm's law. There again, physical studies have shown a variety of boundary layers (see [6]) : $\varepsilon^{1/2}$, $\varepsilon^{1/4}$, $\ldots$.

The rest of this note is structured as follows. Section 2 is devoted to the general presentation of the method. Sections 3 and 4 detail some of its applications, resp. to "flat" and spherical layers and Section 5 raises the problem of stability of boundary layers.

2. Presentation

The first idea is that boundary layers already appear on *linear* equations, therefore in a first step we can dismiss the transport term $(u \cdot \nabla)u$. This is a very classical approach, underlying all the physical studies. This is often *a posteriori* justified, since in many cases the nonlinear term appears to be a higher order perturbation of the linear case. However this nonlinear term is important when we look at stability issues. It is this term which destabilizes many flows and causes the transition of boundary layers from a laminar to a turbulent regime. In a crude way we can say that the nonlinear term does not create the boundary layer, but may destabilize it.

The second idea, once we have a linear equation, is to take the Fourier Laplace transform in space and time. This is of course not new, however a careful analysis of the transform leads to the desired sizes and equations.

Let us formalize a little these ideas. Equations like Navier Stokes Coriolis are of the form

$$\mathcal{A}^\varepsilon U^\varepsilon + \mathcal{Q}^\varepsilon(U^\varepsilon) = F^\varepsilon \tag{8}$$

where for instance $U^\varepsilon = (u^\varepsilon, p^\varepsilon)$, F^ε is some forcing term. In (8), $\mathcal{A}^\varepsilon$ is the linear part of the equation, and $\mathcal{Q}^\varepsilon$ the nonlinear part (in this case $(u^\varepsilon.\nabla)u^\varepsilon$).

The first step is to dismiss the $\mathcal{Q}^\varepsilon$ term and to look at

$$\mathcal{A}^\varepsilon U^\varepsilon = F^\varepsilon. \tag{9}$$

Of course this simplification needs to be justified. A first formal argument is to compute the size of $\mathcal{Q}^\varepsilon(U^\varepsilon)$ for the solution U^ε of (9) and to compare it with $\mathcal{A}^\varepsilon U^\varepsilon$.

The second step is to take Fourier Laplace transform of (9). Let τ and ξ be the dual variables of t and x. Notice that τ and ξ are *complex* numbers. We have

$$A^\varepsilon \mathcal{F} U^\varepsilon = \mathcal{F} F^\varepsilon + \text{boundary and initial terms.} \tag{10}$$

Notice that the boundary and initial terms depend on U^ε itself, therefore (10) does not give simply U^ε as a function of F^ε. These terms combine traces of U^ε and its time and spatial derivatives on $\partial([0 + \infty[\times\Omega)$. In any case, a first step in the study of (10) is to forget these terms and concentrate on

$$A^\varepsilon \mathcal{F} U^\varepsilon = \mathcal{F} F^\varepsilon, \tag{11}$$

an equation which is much more simple than (8)! The main point is that in all the boundary layers we know, the study of (11) is sufficient to get the sizes and the equations of all the layers. This gives a simple way to attack the usual intricate and cumbersome analysis of the boundary layers. To go back to (8) then needs a careful analysis and justification, but on clearly identified equations and layers, which greatly helps ! See section 5 for more details.

Let us go on with the analysis of (11). Let Σ_ε be the characteristic manifold

$$\Sigma_\varepsilon = \{(\tau, \xi, v) \quad | \quad A^\varepsilon(\tau, \xi)v = 0\} \tag{12}$$

and let

$$\sigma_\varepsilon = \{(\tau, \xi) \quad | \quad a^\varepsilon(\tau, \xi) = 0\} \tag{13}$$

where

$$a^\varepsilon = \det A^\varepsilon$$

be its projection in the spectral plane.

Homogeneous solutions of (11) are simply solutions with Fourier transform with support in Σ_ε. A particular solution of (11) reflects the effect of the forcing term on the system. Of course if we take an arbitrary forcing term we can get almost anything, and any type of layers. We have therefore to restrict to physically admissible forcings. A natural one is a smooth forcing term. For instance we may assume that F^ε is smooth, or equivalently that its Fourier transform $\mathcal{F}F^\varepsilon$ is rapidly decreasing in all the variables, or even has compact support, and is independent on ε.

Boundary layers then appear as the reaction of the system at small spatial lengths, or equivalently at large *imaginary* part of ξ. Hence they are described by the behavior of σ_ε for large *complex* ξ. As ε goes to 0, σ_ε may have parts which go to infinity. These parts will describe boundary layers or oscillations (depending whether ξ or τ or both, or only their imaginary parts go to infinity). Therefore the boundary layer sizes are obtained by studying the asymptotic branches Notice that Σ_ε then gives more information, keeping track of the "polarization" of the layer.

Let us be a little more precise. Let us consider the domain $\Omega = \{x_d > 0\}$ with boundary $\partial\Omega = \{x_d = 0\}$. The aim of the game is to find a sequence of homogeneous solutions of the form

$$v^\varepsilon e^{i\tau^\varepsilon t + i\xi^\varepsilon x},$$

that is a sequence $(\tau^\varepsilon, \xi^\varepsilon, v^\varepsilon) \in \Sigma_\varepsilon$ such that $\xi_1, \ldots, \xi_{d-1}$ are independent of ε (Fourier mode in tangential variables) and that one of the following assertions is true

- τ^ε remains bounded and $\mathcal{I}m\, \xi_d^\varepsilon \to \pm\infty$ as $\varepsilon \to 0$. This corresponds to a boundary layer type behavior, with a boundary layer size of order $(\mathcal{I}m\, \xi_d^\varepsilon)^{-1}$.
- ξ_d^ε and $\mathcal{I}m\, \tau^\varepsilon$ remain bounded, and $\mathcal{R}e\, \tau^\varepsilon \to \pm\infty$, which corresponds to an high frequency oscillation in the domain, with frequency of order $\mathcal{R}e\, \tau^\varepsilon$.
- $\mathcal{I}m\, \tau^\varepsilon$ remains bounded, and $\mathcal{R}e\, \tau^\varepsilon \to \pm\infty$ and $\mathcal{I}m\, \xi_d^\varepsilon \to \pm\infty$. This corresponds to an highly time oscillatory boundary layer.
- ξ_d^ε bounded and $\mathcal{I}m\, \tau^\varepsilon \to +\infty$: initial (time) boundary layer.
- $\mathcal{I}m\, \xi_d^\varepsilon \to \pm\infty$ and $\mathcal{I}m\, \tau^\varepsilon \to +\infty$: initial time layer in the boundary layer.

To look for such behaviors then reduces to the search of asymptotic behaviors of solutions of $a^\varepsilon = 0$. Often, a^ε is polynomial. In this case we are left with the problem of finding the possible asymptotics behaviors of roots of polynomials of usually small degree, but with parameters. These behaviors then give the possible sizes of the boundary layers.

The next step is to look for an equation on the boundary layer. Once we have the size, it is already easier. But moreover a look at Σ_ε gives the possible sizes of each of the components (sizes of the velocity field, of the pressure,...). This greatly facilitates the derivation of the boundary layer equation. In fact a more detailed study of $A^\varepsilon U^\varepsilon = 0$ directly gives the desired equation. Let us now detail the method on a few cases (we refer to [1] for an extensive study), namely the classical Ekman layers, vertical rotating layers, MHD layers and a few spherical layers. In this latest case, the arguments are purely formal.

3. First applications

3.1. Rotating fluids

We consider in this section equations (1, 2) of a rotating fluid, with the physical scaling $E = \varepsilon^2$. After we drop the $u \cdot \nabla u$ term, this system can be written.

$$A^\varepsilon U^\varepsilon = 0, \quad U^\varepsilon = \begin{pmatrix} u^\varepsilon \\ p^\varepsilon \end{pmatrix} \tag{14}$$

where $A^\varepsilon = A^\varepsilon(-i\partial_t, -i\partial_{\mathbf{x}})$ is the differential operator with matricial symbol ($\xi^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$) :

$$A^\varepsilon(\tau, \xi) = \begin{pmatrix} i\tau + \varepsilon\xi^2 & -\varepsilon^{-1} & 0 & \varepsilon^{-1}\xi_1 \\ \varepsilon^{-1} & i\tau + \varepsilon\xi^2 & 0 & \varepsilon^{-1}\xi_2 \\ 0 & 0 & i\tau + \varepsilon\xi^2 & \varepsilon^{-1}\xi_3 \\ \varepsilon^{-1}\xi_1 & \varepsilon^{-1}\xi_2 & \varepsilon^{-1}\xi_3 & 0 \end{pmatrix}.$$

The determinant, obtained by expansion along the last row, is

$$a^\varepsilon(\tau, \xi) = (i\tau + \varepsilon\xi^2)^2\xi^2 + \frac{\xi_3^2}{\varepsilon^2}. \tag{15}$$

As explained in section 2., we derive the different boundary layers by looking at elements $(\tau^\varepsilon, \xi^\varepsilon, v^\varepsilon) \in \Sigma_\varepsilon$ satisfying different asymptotics as ε goes to zero. Note that every solution v of $A^\varepsilon(\tau, \xi)v = 0$ satisfies relations

$$v_1 = \frac{-\varepsilon^{-1}\xi_1\gamma + \varepsilon^{-2}\xi_2}{\gamma^2 + \varepsilon^{-2}} v_4, \quad v_2 = \frac{-\varepsilon^{-1}\xi_2\gamma + \varepsilon^{-2}\xi_1}{\gamma^2 + \varepsilon^{-2}} v_4, \quad \gamma v_3 = -\varepsilon^{-1}\xi_3 v_4. \tag{16}$$

3.1.1. Ekman layers

We first derive the possible horizontal boundary layers. They correspond to elements $(\tau^\varepsilon, \xi^\varepsilon, v^\varepsilon) \in \Sigma_\varepsilon$ with ξ_1, ξ_2 independent of ε,

$$\tau^\varepsilon \text{ bounded with } \varepsilon, \mathcal{I}m\, \xi_3^\varepsilon \to +\infty.$$

As $(\tau^\varepsilon, \xi^\varepsilon) \in \sigma_\varepsilon$ we have

$$(i\tau^\varepsilon + \varepsilon(\xi^\varepsilon)^2)^2(\xi^\varepsilon)^2 + \frac{(\xi_3^\varepsilon)^2}{\varepsilon^2} = 0.$$

We deduce as ε goes to zero

$$\varepsilon^2(\xi_3^\varepsilon)^6 \sim -\frac{(\xi_3^\varepsilon)^2}{\varepsilon^2}, \tag{17}$$

$$\Leftrightarrow \quad \xi_3^\varepsilon \sim \frac{\pm 1 \pm i}{\sqrt{2}\varepsilon}, \tag{18}$$

which gives a possible size $O(\varepsilon) = O(E^{1/2})$: that is the so-called "Ekman layer". We now want to derive informations on the "boundary layer solutions" $U^\varepsilon = e^{i\tau^\varepsilon t} e^{i\xi^\varepsilon \mathbf{x}} v^\varepsilon$ of (14), where ξ_3^ε has behaviour given by (18). Namely, we want to know the size of the components of U^ε, and the equation it satisfies at leading order in ε (boundary layer equation). The size of the U^ε components is defined up to a multiplication by an arbitrary function of ε. Using (16), we get

$$v_1^\varepsilon \sim C, \quad v_2^\varepsilon \sim C, \quad v_3^\varepsilon \sim C\varepsilon, \quad v_4^\varepsilon \sim C\varepsilon^2,$$

where C stands for various constants. Back to U^ε we get

$$U^\varepsilon = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon & \\ & & & \varepsilon^2 \end{pmatrix} \left(U^0(\cdot, \varepsilon^{-1}z) + \varepsilon U^1(\cdot, \varepsilon^{-1}z) + \ldots \right).$$

Equation on U^0 is obtained by rescaling the matricial symbol: if we set

$$\underline{A}^\varepsilon(\tau, \xi_1, \xi_2, \xi_3) = A^\varepsilon(\tau, \xi_1, \xi_2, \varepsilon^{-1}\xi_3) \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \varepsilon & \\ & & & \varepsilon^2 \end{pmatrix}$$

we have $\underline{A}^\varepsilon = \underline{A}^0 + \varepsilon \underline{A}^1 + \ldots$, with leading symbol :

$$\underline{A}^0 = \begin{pmatrix} \xi_3^2 & -1 & 0 & 0 \\ 1 & \xi_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix}$$

which corresponds to equations

$$u_2^0 + \partial_Z^2 u_1^0 = 0 \quad (19)$$

$$u_1^0 - \partial_Z^2 u_2^0 = 0 \quad (20)$$

$$\partial_x u_1^0 + \partial_y u_2^0 + \partial_Z u_3^0 = 0. \quad (21)$$

All these results match the derivation obtained in [3] through an asymptotic expansion.

Remark 1 Let us now compute the size of the components of the nonlinear term. We find

$$u.\nabla u_1 = O(1), \quad u.\nabla u_2 = O(1), \quad u.\nabla u_3 = O(\varepsilon).$$

It is small compared to the other terms involved in the equation of the layer (since $-\varepsilon \Delta u = O(\varepsilon^{-1})$), which justifies a posteriori that we neglect the nonlinear terms in the derivation of the equation of the boundary layer. ■

3.1.2. Vertical layers

According to section 2, we look this time for elements $(\tau^\varepsilon, \xi^\varepsilon, v^\varepsilon)$ of Σ^ε with ξ_2, ξ_3 fixed and

$$\tau^\varepsilon \text{ bounded with } \varepsilon, \mathcal{I}m\, \xi_1^\varepsilon \to +\infty.$$

Looking at equation $a^\varepsilon = 0$, there are two cases to consider

- $\xi_3 \neq 0$.

In this case, we get

$$\varepsilon^2 \xi_1^6 \sim -\frac{\xi_3^2}{\varepsilon^2}$$

which shows that $\varepsilon^{2/3}$ is a boundary layer size. Proceeding as above, we get the size of velocity and pressure in the layer

$$u_1 = O(\varepsilon^{2/3}), \quad u_2 = O(1), \quad u_3 = O(1), \quad p = O(\varepsilon^{2/3}).$$

If we rescale these quantities by

$$U^\varepsilon = \begin{pmatrix} \varepsilon^{2/3} & & & \\ & 1 & & \\ & & 1 & \\ & & & \varepsilon^{2/3} \end{pmatrix} \Big(U^0(\cdot, \varepsilon^{-2/3} z) + \varepsilon^{2/3} U^1(..., \varepsilon^{-2/3} z) + ... \Big)$$

equations of the layer are

$$\begin{aligned} u_2^0 - \partial_x p^0 &= 0 \\ u_1^0 + \partial_y p^0 - \partial_x^2 u_2^0 &= 0 \\ \partial_z p^0 + \partial_x^2 u_3^0 &= 0 \\ \partial_x u_1^0 + \partial_y u_2^0 + \partial_z u_3^0 &= 0. \end{aligned}$$

Note that whereas horizontal Ekman layers are described by ordinary differential equations, these vertical Ekman layers lead to a genuine two dimensionnal partial differential equation. It leads to a single equation on pressure

$$\frac{\partial^2 p^0}{\partial z^2} + \frac{\partial^6 p^0}{\partial X^6} = 0 \tag{22}$$

- $\xi_3 = 0$

We get :

$$\xi_1^\varepsilon \sim \pm \sqrt{\frac{\tau^\varepsilon}{\varepsilon}}$$

which depends on time scale τ^ε. If the boundary layer has a "natural" time scale of order 1, the layer has size $\varepsilon^{1/2}$ ($E^{1/4}$), whis is the second scale expected from [8]. Note that $E^{1/4} \sim \nu^{1/2}$ in our case, therefore this layer has the same size as the famous Prandtl layer. As Prandtl layer is highly unstable, it is very likely that this $E^{1/4}$ vertical layer is also linearly and nonlinearly unstable ...

3.2. MHD

We do not detail computations here, and refer to [1] for completeness. Determinant a^ε is given by

$$a^\varepsilon(\tau, \xi) = i\tau + \xi^2 + \frac{\xi_3^2}{\varepsilon \xi^2}$$

which gives

- an horizontal layer with size $\varepsilon^{1/2}$ (horizontal Hartmann layer)
- a vertical layer with size $\varepsilon^{1/4}$

Equations of the horizontal layer are for instance :

$$\begin{aligned} \partial_Z^2 u_1^0 - u_1^0 &= 0 && (23) \\ \partial_Z^2 u_2^0 - u_2^0 &= 0 && (24) \\ \partial_x u_1^0 + \partial_y u_1^0 + \partial_Z u_3^0 &= 0. && (25) \end{aligned}$$

4. Spherical cases

The case of spherical layers is central with regards to its applications in geophysics. In order to derive its main features, we may adapt (in a completely formal way at that time) considerations of section 2.. Let us consider the case of a boundary layer at latitude θ. The natural directions of the problem are those tangent and perpendicular to the sphere, so that we make the change of variables

$$\mathbf{x}' = R_\theta \mathbf{x}, \quad v' = R_\theta v$$

where R_θ is the rotation of angle θ in the (x, z) plane, v being any 3-D vectorial quantity in the equations (fluid velocity, current density, ...). This leads to a new linear system

$$A^{\varepsilon,\theta} V = 0 \tag{26}$$

The main difference with the flat case is the geometrical constraint on the length scales brought by sphericity: if δ (resp. h) is the typical length scale of the layer perpendicularly (resp. tangently) to the sphere, we have $h = O(\sqrt{\delta})$. Following this, we look for solutions $V = e^{i\tau^\varepsilon t} e^{i\xi^\varepsilon \mathbf{x}'} v^\varepsilon$ of (26) with ξ_2 independent of ε, τ^ε bounded with ε, and

$$\mathcal{I}m\, \xi_1^\varepsilon \to +\infty,\ \mathcal{I}m\, \xi_3^\varepsilon \to +\infty,\ |\mathcal{I}m\, \xi_3^\varepsilon| \sim C |\mathcal{I}m\, \xi_1^\varepsilon|^{1/2}. \tag{27}$$

Note that ξ_1^ε corresponds to δ, and goes faster to infinity than ξ_3^ε. Of course all this section is purely formal.

4.1. Rotating fluids

After rotation of angle θ, we get the system

$$A^{\varepsilon,\theta} U' = 0, \quad U' = \begin{pmatrix} u' \\ p' \end{pmatrix},$$

$$A^{\varepsilon,\theta}(\tau, \xi) = \begin{pmatrix} i\tau + \varepsilon\xi^2 & -\varepsilon^{-1}\cos(\theta) & 0 & \varepsilon^{-1}\xi_1 \\ \varepsilon^{-1}\cos(\theta) & i\tau + \varepsilon\xi^2 & \varepsilon^{-1}\sin(\theta) & \varepsilon^{-1}\xi_2 \\ 0 & \varepsilon^{-1}\sin(\theta) & i\tau + \varepsilon\xi^2 & \varepsilon^{-1}\xi_3 \\ \varepsilon^{-1}\xi_1 & \varepsilon^{-1}\xi_2 & \varepsilon^{-1}\xi_3 & 0 \end{pmatrix}.$$

The determinant $A^{\varepsilon,\theta}$ is given by

$$A^{\varepsilon,\theta}(\tau, \xi) = \varepsilon^2 \xi^2 (i\tau + \varepsilon\xi^2)^2 + \sin^2(\theta)\xi_1^2 + \cos^2(\theta)\xi_3^2 - 2\cos(\theta)\sin(\theta)\xi_1\xi_3.$$

We must distinguish two cases

- $\theta \neq 0$ (away from the equator).

We obtain

$$\varepsilon^4 (\xi_1^\varepsilon)^6 \sim -\sin^2(\theta)(\xi_1^\varepsilon)^2$$

which gives a boundary layer with size $|\sin(\theta)|^{-1/2}\varepsilon$. This layer is similar to the flat Ekman layer. Its size increases as θ goes to zero, that is as we get closer to the equator.

- $\theta = 0$.

The above layer degenerates at the equator as $\sin(\theta) = 0$. At $\theta = 0$, the leading terms of $a^{\varepsilon,\theta}$ change and we get

$$\varepsilon^2 (\xi_1^\varepsilon)^4 \sim -(\xi_3^\varepsilon)^2$$

which leads with (27) to the new size $\delta = \varepsilon^{4/5}$. This size matches the results of Stewartson. As in section 3., we get the size of velocity and pressure in the layer at the equator

$$u_1' = O(\varepsilon^{2/5}), \quad u_2' = O(1), \quad u_3' = O(1), \quad p' = O(\varepsilon^{1/5}).$$

After rescaling, equations of the layer are (rescaled variables)

$$\begin{aligned} u_2^0 - \partial_x p^0 &= 0 \\ u_1^0 + \partial_x^2 u_2^0 &= 0 \\ \partial_z p^0 + \partial_x^2 u_3^0 &= 0 \\ \partial_x u_1^0 + \partial_z u_3^0 &= 0 \end{aligned}$$

and it leads again to

$$\partial_Z^2 p^0 + \partial_X^6 p^0 = 0.$$

4.2. MHD

With similar computations, there is a boundary layer with size $\varepsilon^{1/2}$ away from the equator, and $\varepsilon^{2/3}$ at the equator.

5. Some justification

Note that in the derivation of the size of the boundary layer, the nonlinear term does not play anyrole and can be dismissed. For stability issues on the contrary, it plays a central role. Let us go back to the rotating fluids. Let us assume that we are able to construct an approximate solution u^{app} of the form $u^{int} + u^{bl}$ where u^{int} describes its behavior away from the boundaries, and where u^{bl} describes boundary layers. Of course u^{bl} will solve some of the equations derived in the previous section.

The problem is now to know whether "true" solutions u^ε converge to u^{app} as ε goes to 0, assuming for instance that they have the same initial conditions. This problem is a stability question: if u^{app} is a stable solution, then we will get convergence. If on the contrary u^{app} is unstable then u^ε will split from u^{app}, and thus very quickly (typically within times of order $O(\varepsilon \log \varepsilon^{-1})$).

A classical way to investigate the convergence of $v = u^\varepsilon - u^{app}$ is to make an energy estimate on it. For rotating fluids we have

$$\partial_t v + (u^{app} \cdot \nabla) v + (v \cdot \nabla) u^{app} + (v \cdot \nabla) v + \frac{\nabla q}{\varepsilon} - \frac{E}{\varepsilon} \Delta v + \frac{\mathbf{e} \times v}{\varepsilon} = 0,$$

$$\nabla \cdot v = 0.$$

An L^2 energy estimate gives, using divergence free condition,

$$\partial_t \int \frac{v^2}{2} + \int v (v \cdot \nabla) u^{app} + \frac{E}{\varepsilon} \int |\nabla v|^2 = 0.$$

The problem lies in the second term of this identity since ∇u^{app} is very large in the boundary layer (typically of order $O(\varepsilon^{-1})$. Therefore without any further argument we only get

$$\partial_t \int \frac{v^2}{2} + \frac{E}{\varepsilon} \int |\nabla v|^2 \le \frac{C}{\varepsilon} \int \frac{v^2}{2}$$

for some constant C, which is of course not enough to prove convergence. A more detailed analysis then uses Hardy inequality to bound $\int v(v.\nabla) u^{app}$ by $C_0 \varepsilon \int |\nabla v|^2$. This ends the proof provided C_0 is small enough, in order to be absorbed in E/ε^2. But C_0 depends on u^{app} and in fact on $\|u^{int}\|_{L^\infty}$. Therefore we have convergence of $u^\varepsilon - u^{app}$ to 0 under a *smallness* assumption. Physically, with the scaling we assume, $\|u^{int}\|_{L^\infty}$ plays the role of a Reynolds number, and what we mention is a particular case of a basic property of fluid flows: they are stable at low Reynolds numbers and unstable at high Reynolds numbers.

At low Reynolds number, the viscosity dominates the transport term and no instability can occur: the approximate solution is very near the true solution. At high Reynolds numbers on the contrary, the viscosity is no longuer sufficient to control the transport term and instabilities arise, which prevent convergence of the approximate solution to the true solution.

Physically it is therefore expected that there exists some constant C_1 such that if $\|u^{int}\|_{L^\infty} < C_1$ then we have convergence and if $\|u^{int}\|_{L^\infty} > C_1$ we have no convergence. For the time being stability and convergence is proved using L^2 energy methods for $\|u^{int}\|_{L^\infty} \leq C_0 < C_1$ and instability is proved above C_1. The area between C_0 and C_1 is widely open.

References

[1] Gérard-Varet, D. *In preparation.*

[2] Greenspan, H. P. (2000). *The Theory of Rotating Fluids*, Breukelen Press. Brookline, MA.

[3] Grenier, E. and Masmoudi, N. (1997). Ekman layers of rotating fluids, the case of well prepared initial data, *Comm. Partial Differential Equations*, **22**, 953–975.

[4] Pedlosky, J. (1979). *Geophysical Fluid Dynamics*, Springer Verlag.

[5] Proudman, I. (1956). The almost rigid rotation of viscous fluid between concentris spheres, *J. Fluid Mech.*, **1**, 505–516.

[6] Roberts, P. H. (1967). *An introduction to magnetohydrodynamics*, Longmans, Green.

[7] Stewartson, K. (1965). On almost rigid rotations. Part2, *J. Fluid Mech.*, **26**, 131–152.

[8] Stewartson, K. (1975). On almost rigid rotations, *J. Fluid Mech.*, **3**, 17–26.

D. Gérard-Varet, E. Grenier
U.M.P.A., U.M.R. 5669, E.N.S. Lyon
46 allée d'Italie, 69364 Lyon
France