On suprabarrelledness of $c_0(\Omega, X)$

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Abstract. Assuming that $\Omega$ is a non-empty set and $X$ is a real or complex normed space, we show that the linear space $c_0(\Omega, X)$ of all functions $f : \Omega \to X$ such that for each $\varepsilon > 0$ the set $\{ \omega \in \Omega : \| f(\omega) \| > \varepsilon \}$ is finite, endowed with the supremum norm, is suprabarrelled if and only if $X$ is suprabarrelled.

1. Preliminaries

Along this paper $\Omega$ will denote a non-empty set and $X$ a normed space over the field $\mathbb{K}$ of real or complex numbers. We represent by $c_0(\Omega, X)$ the linear space over $\mathbb{K}$ of all those functions $f : \Omega \to X$ such that for each $\varepsilon > 0$ the set $\{ \omega \in \Omega : \| f(\omega) \| > \varepsilon \}$ is finite or empty, equipped with the supremum norm $\| f \|_\infty = \sup \{ \| f(\omega) \| : \omega \in \Omega \}$. Since the support of $f$ is $\bigcup_{n=1}^{\infty} \{ \omega \in \Omega : \| f(\omega) \| > \frac{1}{n} \}$, each $f \in c_0(\Omega, X)$ is countably supported.

If $\Gamma$ is a (possibly empty) subset of $\Omega$, we denote by $c_0(\Gamma, X)$ the linear subspace of $c_0(\Omega, X)$ consisting of all those functions $f$ with $f(\Omega - \Gamma) = \{0\}$.

If $\Omega$ is countable infinite, then we shall write $c_0(X)$ instead of $c_0(\Omega, X)$ ([10]). Hence $c_0(X)$ is the linear space of all sequences in $X$ convergent to zero, endowed with the supremum norm.

Let us recall that a (Hausdorff) locally convex space $E$ is barrelled if each barrel (i.e. each absorbing closed absolutely convex set) in $E$ is a neighbourhood of the origin (see [11], 3.1.2).

An increasing $p$-web in a set $Y$ (see [1]) is a family $W = \{ E_t : t \in T_p \}$, with $T_p = \bigcup_{k=1}^{p} \mathbb{N}^k$, such that $Y = \bigcup_{n \in \mathbb{N}} E_n$, $E_n \subset E_{n+1}$, $E_n = \bigcup_{t \in \mathbb{N}} E_{t,n}$ and $E_{t,n} \subset E_{t,n+1}$, for $t \in T_{p-1}$ and $n \in \mathbb{N}$. If $Y$ is a vector space and every $E_t$ is a linear subspace of $Y$, we say then that $W$ is a linear increasing $p$-web.

A (Hausdorff) locally convex space $E$ is called $p$-barrelled if given in $E$ a linear increasing $p$-web $W = \{ E_t : t \in T_p \}$ there is a $t \in \mathbb{N}^p$ such that $E_t$ is barrelled and dense in $E$ (see [4], [6], [8], [12] and [13]). The 1-barrelled spaces were introduced by Valdivia ([18]) with the name of suprabarrelled spaces, called $(db)$ in [14] and [16].
In [7] it is proved that $c_0(\Omega, X)$ is either barrelled, ultrabornological, or unordered Baire-like (UBL for short, [17]) if and only if $X$ is, respectively, barrelled, ultrabornological or UBL. The case of real or complex continuous functions spaces defined on a locally compact space and vanishing at infinity is considered in [3]. Before [7] there were only a few examples of vector valued functions spaces which were UBL whenever $X$ is (non-complete) UBL. It is natural to ask whether or not the preceding considerations are also true in the class of suprabarrelled spaces, because UBL spaces are suprabarrelled ([6], 3.1 and 3.2.2) and suprabarrelled spaces enjoy useful properties, for instance it is well known that the linear mappings with closed graph from a suprabarrelled space into a (LB)-space have strong localizations properties (see, for instance, [5], [9], [15, 17], [18] and [20]). In this paper we shall prove that $c_0(\Omega, X)$ is suprabarrelled if and only if $X$ is suprabarrelled. Also a new property about linear increasing 1–webs in $c_0(\Omega, X)$ will be obtained. In what follows $\langle V \rangle$ means the linear hull of $V$. We are going to use the classical notation given, for instance, in [2] and [19].

2. Suprabarrelledness in $c_0(\Omega, X)$

Let us suppose that $c_0(\Omega, X)$ is the union of an increasing sequence $\{F_n\}_{n=1}^{\infty}$ of subspaces. Let $T_n$ be a barrel in $F_n$, $V_n = T_n(\Omega, X)$, $Z_n = \langle V_n \rangle$ and $S_n = \bigcap \{Z_m : m \geq n\}$. From $F_n \subset S_n$ it follows that $c_0(\Omega, X) = \bigcup_{n=1}^{\infty} S_n$.

Lemma 1 If $F$ is a suprabarrelled subspace of $c_0(\Omega, X)$ there exists $n \in \mathbb{N}$ such that $F \subset S_n$.

Proof. \{ $F \cap S_n : n \in \mathbb{N}$ \} is an increasing covering of $F$. The suprabarrelledness implies that there is an $F \cap S_n$ which is barrelled and dense in $F$. Then, if $n \geq m$ we have that $T_m^{c_0(\Omega, X)}$ contains a neighbourhood of 0 in $F \cap S_n$. By density we have that $T_m^{c_0(\Omega, X)}$ also contains a neighbourhood of 0 in $F$. This implies that $F \subset Z_m$ when $m \geq n$. It follows that $F \subset \bigcap \{Z_m : m \geq n\} = S_n$.

The preceding lemma has the following obvious extension which will be useful in the end of Proposition 1.

Lemma 2 Let $F$ be a subspace of $c_0(\Omega, X)$ and $\mathcal{T}$ a locally convex topology in $F$ finer than the induced by $c_0(\Omega, X)$. If $(F, \mathcal{T})$ is suprabarrelled there exists $n \in \mathbb{N}$ such that $F \subset S_n$.

Proof. Since \{ $F \cap S_n : n \in \mathbb{N}$ \} is an increasing covering of $(F, \mathcal{T})$ there exists an $(F \cap S_n, \mathcal{T}_F \cap S_n)$ which is barrelled and dense in $(F, \mathcal{T})$. If $n \leq m$, then $F \cap S_n \cap T_m^{c_0(\Omega, X)}$ is a barrel in $F \cap S_n$ endowed with the topology induced by $\mathcal{T}$. Therefore $T_m^{c_0(\Omega, X)}$ contains a neighbourhood of 0 in $(F, \mathcal{T})$, implying that $F \subset Z_m$ when $m \geq n$. Then $F \subset \bigcap \{Z_m : m \geq n\} = S_n$.

Proposition 1 There exists a finite set $\Delta$ (possibly empty) and a natural $n$ such that $c_0(\Omega \setminus \Delta, X) \subset S_n$.

Proof. First step: We are going to prove that there exists a countable set $\Delta$ and a natural number $n$ such that $c_0(\Omega \setminus \Delta, X) \subset S_n$. If this were not true, there would be a $f_1 \in c_0(\Omega, X)$ with $\|f_1\|_{\infty} = 1$ and $f_1 \notin S_1$.

The set $\Delta_1 = \text{supp}(f_1)$ is countable and from $c_0(\Omega \setminus \Delta_1, X) \notin S_2$ we deduce the existence of a $f_2 \in c_0(\Omega \setminus \Delta_1, X)$ with $\|f_2\|_{\infty} = 1$ and $f_2 \notin S_2$. The set $\Delta_2 = \text{supp}(f_2)$ is countable and we choose $f_3 \in c_0(\Omega \setminus \{\Delta_1 \cup \Delta_2\}, X)$ with $\|f_3\|_{\infty} = 1$ and $f_3 \notin S_3$.

By induction we would obtain a bounded sequence \{ $f_n : n \in \mathbb{N}$ \} in $c_0(\Omega, X)$ and a pairwise disjoint sequence \{ $\Delta_n : n \in \mathbb{N}$ \} of countable subsets of $\Omega$ such that $\Delta_n = \text{supp}(f_n)$, $\|f_n\|_{\infty} = 1$ and $f_n \notin S_n$ for each $n \in \mathbb{N}$.

The mapping $\varphi$ from $c_0$ into $c_0(\Omega, X)$ defined by $\varphi (\{\xi_n : n \in \mathbb{N}\}) = \sum_{n=1}^{\infty} \xi_n f_n$ is well-defined since $\{\xi_n : n \in \mathbb{N}\} \in c_0$, $f_n : n \in \mathbb{N}$ is bounded and for $\omega \in \Omega$ we have that $\sum_{n=1}^{\infty} \xi_n f_n(\omega)$ has at most one
non-null term. It is also obvious that $\varphi$ is an isometry onto and then, by Lemma 1, we have that there exists an $n \in \mathbb{N}$ such that $\varphi (c_0) \subset S_n$. Then the relation $f_n \in \varphi (c_0) \subset S_n$ contradicts the choice $f_n \notin S_n$ and with this contradiction concludes the first part of the proof. Without loss of generality we may suppose that $\Delta = \mathbb{N}$.

Second step: We are going to prove that there is a natural number $i$ such that $c_0(\mathbb{N}\setminus \{1, 2, 3, \ldots, i\}, X) \subset S_i$.

In fact, if $c_0(\mathbb{N}\setminus \{1, 2, 3, \ldots, i\}, X) \not\subset S_i$, for $i = 1, 2, 3, \ldots$ then there exists a sequence $\{f_i : i \in \mathbb{N}\}$ such that $f_i \in c_0(\mathbb{N}\setminus \{1, 2, 3, \ldots\}, X) - S_i$ and $\|f_i\|_{\infty} = 1$ for $i = 1, 2, 3, \ldots$. Then the mapping $\varphi$ from $l_1$ into $c_0(\mathbb{N}, X)$ defined by

$$\varphi (\{\xi_n : n \in \mathbb{N}\}) = \sum_{n=1}^{\infty} \xi_n f_n$$

is well-defined since $\sum_{n=1}^{\infty} \xi_n f_n(i)$ has at most $i - 1$ non-null terms, and $\sum_{n=1}^{\infty} \xi_n f_n \in c_0(\mathbb{N}, X)$ because if $\varepsilon > 0$ there is a $k$ such that $\sum_{n=k}^{\infty} |\xi_n| < \frac{\varepsilon}{2}$, implying that $\sum_{n=k}^{\infty} \xi_n f_n \in c_0(\mathbb{N}, X)$ there exists a $p \in \mathbb{N}$ such that $\sum_{n=1}^{k-1} |\xi_n f_n(i)| < \frac{\varepsilon}{2}$ when $i \geq p$. Therefore $\sum_{n=1}^{\infty} |\xi_n f_n(i)| \leq \sum_{n=1}^{k-1} |\xi_n f_n(i)| + \sum_{n=k}^{\infty} |\xi_n f_n(i)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ for $i \geq p$, which proves that $\sum_{n=1}^{\infty} \xi_n f_n \in c_0(\mathbb{N}, X)$.

If $\xi \in l_1$ we have that $\|\varphi (\xi)\| \leq \|\xi\|_1$. The continuity of $\varphi$ enables us to consider in $\varphi (l_1)$ the finest locally convex topology $T$ such that $\varphi : l_1 \to (\varphi (l_1), T)$ is continuous. Then $T$ is finer than the topology induced by $c_0(\mathbb{N}, X)$ and $(\varphi (l_1), T)$ is superbarrelled because it is isometric to a quotient of $l_1$. Then, by Lemma 2, there exists an $n \in \mathbb{N}$ such that $\varphi (l_1) \subset S_n$, giving the contradiction $f_n \in \varphi (l_1) \subset S_n$. This establishes this second step.

The two steps give directly the proposition. ■

**Theorem 1** If $X$ is superbarrelled there is a natural $n$ such that $c_0(\Omega, X) = S_n$.

**Proof.** By the preceding proposition we only need to prove that if $\Delta$ is a finite set there is a $n \in \mathbb{N}$ such that $c_0(\Delta, X) \subset S_n$. This follows from the Lemma 1, the isomorphism $c_0(\Delta, X) = X^\Delta$ and the fact that the product of superbarrelled spaces is superbarrelled (see Proposition 3.2.10 in [6]). ■

**Theorem 2** $X$ is superbarrelled if and only if $c_0(\Omega, X)$ is superbarrelled, being $\Omega$ a non-void set.

**Proof.** For $p \in \Omega$, the spaces $X$ and the quotient $c_0(\Omega, X)/c_0(\Omega \setminus \{p\}, X)$ are isometric. Then, if $c_0(\Omega, X)$ is superbarrelled we have that $X$ is superbarrelled by Proposition 3.2.12 in [6].

Conversely, if $X$ is superbarrelled we have that $c_0(\Omega, X)$ is Baire-like [15] by Proposition 2.2 in [7] and Proposition 1.2.1 in [6]. Then, if $\{F_n : n \in \mathbb{N}\}$ is a linear increasing 1-web of $c_0(\Omega, X)$ there is an $n \in \mathbb{N}$ such that $F_n$ is dense in $c_0(\Omega, X)$ for $m \geq p$.

Therefore, if $c_0(\Omega, X)$ were not superbarrelled we could find an increasing covering $\{F_n : n \in \mathbb{N}\}$ of $c_0(\Omega, X)$, such that each $F_n$ is non-barrelled and dense in $c_0(\Omega, X)$. Let $T_n$ be a barrel in $F_n$ which is not neighbourhood of 0 in $F_n$. If $V_n = T_n^{-c_0(\Omega, X)}$ and $S_n = \bigcap_{m \geq n} (V_m)$ we have by Theorem 1 that there is an $n$ such that $c_0(\Omega, X) = S_n$.

Then $S_n = \langle V_n \rangle$ and, by Proposition 2.2 in [7], we have that $V_n$ is a neighbourhood of 0 in $S_n$, implying that $T_n$ is a neighbourhood of 0 in $F_n$. This contradiction proves the theorem. ■

**References**


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