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A general approximation theorem of Whitney type

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Dedicated to the memory of Klaus Floret

Abstract. We show that Whitney's approximation theorem holds in a general setting including spaces of (ultra)differentiable functions and ultradistributions. This is used to obtain real analytic modifications for differentiable functions including optimal estimates. Finally, a surjectivity criterion for continuous linear operators between Fréchet sheaves is deduced, which can be applied to the boundary value problem for holomorphic functions and to convolution operators in spaces of ultradifferentiable functions and ultradistributions.

Un teorema de aproximación general de tipo Whitney

Resumen. Probamos que el teorema de aproximación de Whitney se cumple en un contexto general que incluye espacios de funciones (ultra)diferenciables y de ultradistribuciones. Este resultado se usa para obtener modificaciones real analíticas de funciones diferenciables incluyendo estimaciones óptimas. Finalmente se deduce un criterio para la sobreyectividad de operadores lineales y continuos entre haces de Fréchet que puede ser aplicado a problemas de valores frontera de funciones holomorfas y a operadores de convolución entre espacios de funciones ultradiferenciables y de ultradistribuciones.

1. Introduction

Whitney's approximation theorem roughly states that C^k-functions defined on an open set $\Omega \subset \mathbb{R}^n$ can be approximated by real analytic functions with arbitrary precision and up to any order of derivatives. Whitney used this result in his famous paper [24] to show that a C^k-Whitney jet f defined on a closed set $F \subset \mathbb{R}^n$ can be extended to a function $\widetilde{f} \in C^k(\mathbb{R}^n)$ such that $\widetilde{f}|_\Omega \in A(\Omega)$, that is, $\widetilde{f}|_\Omega$ is real analytic on $\Omega := \mathbb{R}^n \backslash F$.

For C^∞-functions, Whitney's approximation theorem can be stated as follows:

Theorem 1 ([24, Lemma 6]) *Let $\Omega \subset \mathbb{R}^n$ be open and let η: $\Omega \longrightarrow]0, \infty[$ be continuous. For any $f \in C^\infty(\Omega)$ there is $g \in A(\Omega)$ such that*

$$\left| f^{(a)}(x) - g^{(a)}(x) \right| \leq \eta(x) \text{ if } x \in \Omega \text{ and } |a| \leq 1/\eta(x) . \quad \blacksquare$$

A corresponding result also holds for functions in $C^k(\Omega)$.

In this paper we will show that Whitney's approximation theorem holds in a rather general setting including spaces of (ultra)differentiable functions, locally integrable functions and also ultradistributions.

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In fact, the key result can be stated as a decomposition lemma (see section 2) having a surprisingly short functional analytic proof based on a surjectivity criterion for continuous linear mappings between Fréchet spaces (see Meise, Vogt [14, 26.1]).

We then deduce Whitney's approximation theorem for several concrete spaces from analysis. This is applied to Whitney's extension problem mentioned above using Ω-modification operators as they were studied in Langenbruch [12].

In section 5 we finally prove a surjectivity criterion related to Whitney's approximation theorem (see Theorem 6), which roughly states that a continuous linear operator between Fréchet sheaves is surjective if its range contains the sections with compact support and the real analytic functions. This is then applied to the representation of ultradistributions as boundary values of holomorphic functions and to the surjectivity of convolution operators in spaces of ultradifferentiable functions and ultradistributions, respectively.

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2. A decomposition lemma

In this section we will prove a basic decomposition lemma, which can be considered as a general version of Whitney's approximation theorem (see the discussion in section 3).

We use ultradifferentiable functions of Beurling type as a general frame for our considerations. The reader is referred to Komatsu [9] for further information. In this paper, $(M_p)_{p\in\mathbb{N}_0}$ always is a sequence of positive numbers satisfying the following two standard assumptions

$$m_p := M_{p+1}/M_p \text{ is increasing} \tag{M1}$$

and

$$\sum_p 1/m_p < \infty\,. \tag{M3$'$}$$

For $\Omega \subset \mathbb{R}^n$ open let

$$\mathfrak{E}_{(M_p)}(\Omega) := \Big\{ f \in C^\infty(\Omega) \Big|\, \forall\, K \subset\subset \Omega\, \forall\, C > 0 :$$
$$\|f\|_{K,C} := \sup\big\{ \big| f^{(\alpha)}(x) \big| / \big(M_{|\alpha|} C^{|\alpha|}\big) \big|\, x \in K,\, \alpha \in \mathbb{N}_0^n \big\} < \infty \Big\}$$

and

$$D_{(M_p)}(\Omega) := \big\{ f \in \mathfrak{E}_{(M_p)}(\Omega) \;\big|\; \operatorname{supp} f \subset\subset \Omega \big\}$$

be endowed with their natural topologies. $\mathfrak{E}_{(M_p)}(\Omega)$ is the space of ultradifferentiable functions of Beurling type.

The assumptions needed for decomposition are summarized in the following definition:

Definition 1 *Let $\Omega \subset \mathbb{R}^n$ be open. Two Fréchet spaces $\big(F(\Omega),\, E(\Omega)\big)$ are called a decomposition pair if there is a sequence $(M_p)_p$ as above such that*

i) $\mathfrak{E}_{(M_p)}(\Omega) \subset E(\Omega)$ *is dense and*

$$\mathrm{id} : \mathfrak{E}_{(M_p)}(\Omega) \longrightarrow E(\Omega) \text{ is continuous}\,.$$

ii) $D_{(M_p)}(\Omega) \subset F(\Omega) \subset E(\Omega)$ *and*

$$\mathrm{id} : F(\Omega) \longrightarrow E(\Omega) \text{ is continuous}\,.$$

iii) Let τ_E and τ_F be the topologies of $E(\Omega)$ and $F(\Omega)$, respectively. Then the mapping

$$M_\varphi : \big(D_{(M_p)}(\Omega), \tau_E\big) \longrightarrow \big(D_{(M_p)}(\Omega),\, \tau_F\big),\; f \longrightarrow \varphi f\,,$$

is continuous for $\varphi \in D_{(M_p)}(\Omega)$.

By Definition 1 i),

$$D_{(M_p)}(\Omega) \text{ is dense in } E(\Omega)\,. \tag{1}$$

The mapping M_φ from Definition 1 iii) thus can uniquely be extended to a continuous linear mapping

$$M_\varphi : E(\Omega) \longrightarrow F(\Omega)\,. \tag{2}$$

For $\Omega \subset \mathbb{R}^n$ open and $\varepsilon > 0$ let

$$\Omega^* := \big\{ z \in \mathbb{C}^n \;\big|\; \operatorname{Re} z \in \Omega,\ |\operatorname{Im} z| < \operatorname{dist}(\operatorname{Re} z, \partial\Omega) \big\} \text{ and}$$

$$K_\varepsilon := \big\{ x \in \Omega \;\big|\; \operatorname{dist}(x, \partial\Omega) \geq \varepsilon \text{ and } |x| \leq 1/\varepsilon \big\}\,,$$

and let $K^*_\varepsilon := (K_\varepsilon)^*$ be defined similarly. Let $H(\Omega^*)$ denote the holomorphic functions on Ω^* endowed with the seminorms

$$|f|_{K^*_\varepsilon} := \sup\big\{ |f\,(z)| \;\big|\; z \in K^*_\varepsilon \big\}\,.$$

Lemma 1 *Let $\big(F(\Omega), E(\Omega)\big)$ be a decomposition pair. Then the mapping*

$$A : H(\Omega^*) \times F(\Omega) \longrightarrow E(\Omega),\; A(f,g) := f|_\Omega + g\,,$$

is surjective.

PROOF. A is defined and continuous by Definition 1 i) and ii). Since all spaces involved are Fréchet spaces, we have to show by Meise, Vogt [14, 26.1] that $B \subset E(\Omega)'_b$ is bounded if

$$B|_{H(\Omega^*)} \subset H(\Omega^*)'_b \text{ is bounded and} \tag{3}$$

$$B|_{F(\Omega)} \subset F(\Omega)'_b \text{ is bounded}\,. \tag{4}$$

By (3) there are $\varepsilon > 0$ and $C \geq 1$ such that

$$\big|T(g)\big| \leq C|g|_{K^*_{2\varepsilon}} \text{ if } T \in B \text{ and } g \in H(\Omega^*)\,. \tag{5}$$

Let $f_k\,(z) := (k/\pi)^{n/2} \exp\big(-k\sum_{j=1}^n z_j^2\big)$ and $\varphi \in D_{(M_p)}(\Omega\backslash K_{2\varepsilon})$. It is easily seen that

$$\varphi * f_k \longrightarrow \varphi \text{ in } \mathfrak{E}_{(M_p)}(\mathbb{R}^n) \text{ and } \big|\varphi * f_k\big|_{K^*_{2\varepsilon}} \longrightarrow 0\,. \tag{6}$$

We thus get for $T \in B$ and $\varphi \in D_{(M_p)}\big(\Omega\backslash K_{2\varepsilon}\big)$ by Definition 1 i), (5) and (6)

$$\big|T(\varphi)\big| = \lim_k \big|T(\varphi * f_k)\big| \leq C \lim_k \big|\varphi * f_k\big|_{K^*_{2\varepsilon}} = 0\,. \tag{7}$$

Choose $\psi \in D_{(M_p)}(K_\varepsilon)$ such that $\psi = 1$ near $K_{2\varepsilon}$. Then (1) implies that

$$T = T|_{F(\Omega)} \circ M_\psi \text{ if } T \in B \tag{8}$$

since this holds on $D_{(M_p)}(\Omega)$ by (7). B is thus bounded in $E(\Omega)'_b$ by (8), (2) and (4). ■

Remark 1 Since the Gauss - Kernel is real on $\mathbb{R}^n$, the Decomposition Lemma also holds for real decomposition pairs $\big(E(\Omega), F(\Omega)\big)$ with $H(\Omega^*)$ substituted by

$$H_{\mathbb{R}}(\Omega^*) := \{f \in H(\Omega^*) \mid f\,|_\Omega \text{ is real valued }\}. \quad \blacksquare$$

The Decomposition Lemma will be applied in the next sections in several situations.

The holomorphic part in the Decomposition Lemma is defined on Ω^* since we used the Gauss-kernels f_k in (6). In the case of one variable, this can be improved considerably:

Theorem 2 *Let $\Omega \subset \mathbb{R}$ be open and let $\big(F(\Omega), E(\Omega)\big)$ be a decomposition pair. Then the mapping*

$$A : H\big(\mathbb{C}\backslash\partial\Omega\big)\times F(\Omega) \longrightarrow E(\Omega),\ A(f,g) := f|_\Omega + g\,.$$

is surjective.

PROOF. The theorem holds for $\Omega = \mathbb{R}$ by the Decomposition Lemma. So let $\Omega \neq \mathbb{R}$ and thus $\partial\Omega \neq \varnothing$. Similarly as above we have to show that $B \subset E(\Omega)'_b$ is bounded if

$$B|_{H(\mathbb{C}\backslash\partial\Omega)} \subset H(\mathbb{C}\backslash\partial\Omega)'_b \text{ is bounded and} \tag{9}$$

$$B|_{F(\Omega)} \subset F(\Omega)'_b \text{ is bounded}\,. \tag{10}$$

By (9) there are $\varepsilon > 0$ and $C \geq 1$ such that

$$\big|T(g)\big| \leq C|g|_{\widetilde{K}_{3\varepsilon}} \text{ if } T \in B \text{ and } g \in H\big(\mathbb{C}\backslash\partial\Omega\big) \tag{11}$$

where $\widetilde{K}_{3\varepsilon} := \big\{z \in \mathbb{C}\backslash\partial\Omega \;\big|\; |z|_1 \leq 1/(3\varepsilon) \text{ and } \operatorname{dist}(z,\partial\Omega) \geq 3\varepsilon\big\}$.

Let $\varphi \in D_{(M_p)}\big(\Omega\backslash K_{2\varepsilon}\big)$. We want to show that $T(\varphi) = 0$ if $T \in B$. The support of φ has finitely many components and we may assume that $\operatorname{supp}\varphi$ is contained in a component $I :=]a,b[$ of Ω with $-\infty \leq a < b < \infty$ (or $-\infty < a < b \leq \infty$). Let $a = -\infty$ and $b < \infty$ (The cases $-\infty < a < b \leq \infty$ are treated similarly). We can then assume that $\operatorname{supp}\varphi \subset]-\infty, 1/2\varepsilon[$ or $\operatorname{supp}\varphi \subset]b-2\varepsilon, b[$. The first case is treated as in the proof of the Decomposition Lemma. To consider the second, we use the shifted Cauchy kernels $h_k(z) := 1/\big(2\pi i(z - i/k)\big)$ and set $\varphi_k := (h_k - h_{-k}) * (\varphi \otimes \delta)$. Then φ_k is holomorphic on $\mathbb{C}\backslash\big([b-2\varepsilon, b]\big)\times\{-1/k, 1/k\}$. By Runge's theorem, there is a sequence $f_j \in H\big(\mathbb{C}\backslash\{b\}\big)$ such that $f_j \longrightarrow \varphi_k$ uniformly on compact subsets of $\mathbb{C}\backslash J_k$, where $J_k := \big([b-2\varepsilon, b]\times\{-1/k, 1/k\} \cup \big(\{b\}\times[-1/k, 1/k]\big)$. Hence, $f_j \longrightarrow \varphi_k$ also in $\mathfrak{E}_{(M_p)}(\Omega)$ and we get by Definition 1i) and (11)

$$\big|T(\varphi_k)\big| = \lim_j \big|T(f_j)\big| \leq C\lim_j |f_j|_{\widetilde{K}_{3\varepsilon}} = C|\varphi_k|_{\widetilde{K}_{3\varepsilon}} \text{ if } T \in B \text{ and } k \geq 1/\varepsilon\,.$$

$\varphi_k \longrightarrow 0$ uniformly on $\widetilde{K}_{3\varepsilon}$ and $\varphi_k \longrightarrow \varphi$ in $\mathfrak{E}_{(M_p)}(\mathbb{R})$, since the sequence converges in $C(\mathbb{R})$ by the theorem of dominated convergence and since $\partial_x^\alpha\varphi_k = \big(h_k - h_{-k}\big)*\big(\partial_x^\alpha\varphi \otimes \delta\big)$. We conclude that $T(\varphi) = \lim_k T(\varphi_k) = 0$, and the proof is completed as for the Decomposition Lemma. $\blacksquare$

3. Approximation theorems of Whitney type

In this section we will apply the Decomposition Lemma to deduce approximation results of Whitney type in several function spaces including the space $\big(D_{\{M_p\}}(\Omega)\big)'_b$ of ultradistributions of Roumieu type (see Example 1e)). Here and in the following E'_b denotes the strong dual of a locally convex space E.

Let $(M_p)_p$ always be a sequence of positive numbers satisfying (M1) and (M3′) and let $\|\ \|_\Omega$ denote the sup-norm on Ω.

Example 1 *Let $\eta : \Omega \longrightarrow]0,1/2]$ be continuous. The following spaces $\big(F(\Omega), E(\Omega)\big)$ are decomposition pairs:*

a) $E(\Omega) := \mathfrak{E}_{(M_p)}(\Omega)$ *and* $F(\Omega) := \mathfrak{E}_{(M_p)}(\Omega,\eta) :=$

$$\big\{ f \in C^\infty(\Omega) \;\big|\; p_k(f) := \sup_a \big(\|f^{(a)}/\eta^{k(1+|a|)}\|_\Omega / M_{|a|} \big) < \infty \text{ if } k \in \mathbb{N} \big\}$$

b) $E(\Omega) := C^\infty(\Omega)$ *and* $F(\Omega) := C^\infty(\Omega,\eta) :=$

$$\Big\{ f \in C^\infty(\Omega) \;\big|\; p_k(f) := \sup\big\{ \|f^{(a)}/\eta^k\|_\Omega \;\big|\; |a| \le k/\eta\,(x) \big\} < \infty \text{ if } k \in \mathbb{N} \Big\}$$

c) For $k \in \mathbb{N}_0$ *let* $E(\Omega) := C^k(\Omega)$ *and* $F(\Omega) := C^k(\Omega,\eta) :=$

$$\Big\{ f \in C^k(\Omega) \;\big|\; p_1(f) := \sup\big\{ \|f^{(a)}/\eta\|_\Omega \;\big|\; |a| \le k \big\} < \infty \Big\}$$

d) For $1 \le p < \infty$ *let* $E(\Omega) := L_p^{\text{loc}}(\Omega)$ *and*

$$F(\Omega) := L_p(\Omega,\eta) := \big\{ f \in L_p(\Omega) \;\big|\; p_1(f) := \|f/\eta\|_{L_p(\Omega)} < \infty \big\}$$

e) $E(\Omega) := \big(D_{\{M_p\}}(\Omega,\eta)\big)_b'$ *and* $F(\Omega) := \big(\mathfrak{E}_{\{M_p\}}(\Omega,\eta)\big)_b'$ *where*

$$D_{\{M_p\}}(\Omega) := \big\{ f \in C_0^\infty(\Omega) \;\big|\; \exists\, k \ge 1 : \|f\|_{\mathbb{R}^n,k} < \infty \big\} \text{ and}$$
$$\mathfrak{E}_{\{M_p\}}(\Omega,\eta) := \Big\{ f \in C^\infty(\Omega) \;\big|$$
$$\exists\, k \in \mathbb{N} : q_k\,(f) := \sup\big\{ \big| f^{(a)}\,(x)\,\eta\,(x)^{\,k(1+|a|)} \big| / M_{|a|} \;\big|\; x \in \Omega, a \in \mathbb{N}_0^n \big\} < \infty \Big\}.$$

PROOF. The conditions in Definition 1 are trivially satisfied in the cases a) - d) (in case a), Definition 1iii) is contained in Komatsu [9, Theorem 2.8]). In case e), $E(\Omega)$ and $F(\Omega)$ are Fréchet spaces, since the spaces $D_{\{M_p\}}(\Omega)$ and $\mathfrak{E}_{\{M_p\}}(\Omega,\eta)$ of test functions are compact injective limits of Banach spaces, hence (DFS)-spaces. To check Definition 1 we notice that i) is well-known, ii) holds since $D_{\{M_p\}}(\Omega)$ is continuously embedded and dense in $\mathfrak{E}_{\{M_p\}}(\Omega,\eta)$, and iii) holds since

$$M_\varphi : \mathfrak{E}_{\{M_p\}}(\Omega,\eta) \longrightarrow D_{\{M_p\}}(\Omega) \text{ is continuous.} \quad \blacksquare$$

$\big(\mathfrak{E}_{\{M_p\}}(\Omega,\eta)\big)_b'$ is a weighted space of ultradistributions of Roumieu type endowed with its canonical dual norms

$$p_k\,(f) := q_k^*\,(f) := \sup\big\{ \big|\langle f,\psi\rangle\big| \;\big|\; \psi \in \mathfrak{E}_{\{M_p\}}(\Omega,\eta), q_k(\psi) \le 1 \big\}\,. \tag{12}$$

b) and c) of the following approximation theorem are Whitney's results mentioned in the introduction. The proof of the theorem via the Decomposition Lemma 1 is almost trivial.

Theorem 3 *Let $\eta : \Omega \longrightarrow]0,\infty[$ be continuous.*

a) For any $f \in \mathfrak{E}_{(M_p)}(\Omega)$ *there is* $g \in H(\Omega^*)$ *such that*

$$\big| f^{(a)}\,(x) - g^{(a)}\,(x) \big| \le \eta\,(x)^{\,1+|a|} M_{|a|} \text{ if } x \in \Omega \text{ and } a \in \mathbb{N}_0^n$$

b) For any $f \in C^\infty(\Omega)$ *there is* $g \in H(\Omega^*)$ *such that*

$$\big| f^{(a)}\,(x) - g^{(a)}\,(x) \big| \le \eta\,(x) \text{ if } x \in \Omega \text{ and } |a| \le 1/\eta\,(x)\,.$$

c) Let $k \in \mathbb{N}_0$. For any $f \in C^k(\Omega)$ there is $g \in H(\Omega^*)$ such that

$$\left|f^{(a)}(x) - g^{(a)}(x)\right| \leq \eta(x) \text{ if } x \in \Omega \text{ and } |a| \leq k\,.$$

d) For any $f \in L_p^{\mathrm{loc}}(\Omega)$, $1 \leq p < \infty$, there is $g \in H(\Omega^*)$ such that

$$\left\|(f-g)/\eta\right\|_{L_p(\Omega)} \leq 1$$

e) For any $f \in \left(D_{\{M_p\}}(\Omega)\right)'_b$ there is $g \in H(\Omega^*)$ such that for any $\psi \in D_{\{M_p\}}(\Omega)$

$$\left|\langle f-g,\psi\rangle\right| \leq \sup\left\{\left|\psi^{(a)}(x)\,\eta(x)^{\,1+|a|}\right|/M_{|a|} \;\middle|\; x\in\Omega,\, a\in\mathbb{N}_0^n\right\}.$$

PROOF. We may assume that $\eta(\Omega) \subset \,]0,1/2[$ and apply the Decomposition Lemma 1 to the Examples 1 a) – e). The mapping A is open since it is surjective and the spaces involved are Fréchet spaces. Hence, there are a continuous seminorm $\|\ \|$ on $E(\Omega)$ and $C \geq 1$ such that for any $f \in E(\Omega)$ there is $g \in H(\Omega^*)$ such that

$$p_1(f-g) \leq C\|f\|\,. \tag{13}$$

Since $H(\Omega^*)$ is dense in $E(\Omega)$ (e.g. by Definition 1i) and (6)), there is $g_1 \in H(\Omega^*)$ such that $\|f - g_1\| \leq 1/C$. With $g_2 \in H(\Omega^*)$ chosen for $(f - g_1)$ by (13), we get for $g := g_1 - g_2$

$$p_1(f-g) \leq C\|f-g_1\| \leq 1\,.$$

This shows the claim in any of the five cases, where in case e) the definition (12) of the norm p_1 in $\mathfrak{E}_{\{M_p\}}(\Omega,\eta)'_b$ is used. ■

The Approximation Theorem 3a), d) and e) provides a version of Whitney's approximation theorem for ultradifferentiable and L_p^{loc}-functions and for the space $D_{\{M_p\}}(\Omega)'_b$ of ultradistributions of Roumieu type, respectively.

Using Theorem 2 (and Runge's theorem) instead of the Decomposition Lemma, we can substitute $H(\Omega^*)$ by $H(\mathbb{C}\backslash\partial\Omega)$ in the Approximation Theorem 3 if $\Omega \subset \mathbb{R}$.

4. Whitney's extension theorem

We already mentioned in the introduction that Whitney's approximation theorem is connected to Whitney's extension theorem. In fact, Whitney's approximation theorem is used to extend Whitney jets g defined on F such that the extensions are analytic on Ω. Here and in the following

$$F \subset \mathbb{R}^n \text{ is always closed and } \Omega := \mathbb{R}^n\backslash F\,.$$

This "analytic extension" may be obtained in two steps: first, g is extended to a function $f \in C^\infty(\mathbb{R}^n)$ and then an Ω-modification in $C^\infty(\mathbb{R}^n)$ is chosen for f in the sense of the following

Definition 2 *Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and $f \in C^m(\mathbb{R}^n)$. A function $\widetilde{f}$ is called an Ω-modification in $C^m(\mathbb{R}^n)$ for f if $\widetilde{f} \in C^m(\mathbb{R}^n)$ and*

$$\widetilde{f} \text{ is real analytic on } \Omega \tag{14}$$

$$\partial^\alpha \widetilde{f}\big|_F = \partial^\alpha f\big|_F \text{ for any } \alpha \in \mathbb{N}_0^n \text{ with } |\alpha| \leq m. \tag{15}$$

Ω-modifications and corresponding continuous linear Ω-modification operators have been studied by Valdivia [21, 22], Schmets and Valdivia ([17], [18], [19]) and Brück, Frerick [4] for differentiable and ultradifferentiable functions. Using boundary values of harmonic functions, we obtained in Langenbruch [12] a unified short proof of these results and an explicit formula for a continuous linear Ω-modification operator on

$$\mathfrak{BC}^{\infty}(\mathbb{R}^n) := \left\{ f \in C^{\infty}(\mathbb{R}^n) \;\middle|\; f^{(a)} \text{ is bounded for } a \in \mathbb{N}_0^n \right\},$$

whose restriction to classes of ultradifferentiable functions is also a continuous modification operator within these classes.

To be precise, the following basic result was shown in Langenbruch [12, Theorem 1.2]: Assume that there is $h \in \mathfrak{BC}^{\infty}(\mathbb{R}^n)$ such that

$$h\,|_{\Omega} \text{ is real analytic}, 0 < h\,|_{\Omega} \leq 1 \text{ and } \partial^{\alpha} h_{\big|F} = 0 \text{ for any } \alpha \in \mathbb{N}_0^n\,. \tag{16}$$

For $f \in \mathfrak{BC}^{\infty}(\mathbb{R}^n)$ let $T_h(f)$ be defined by

$$T_h(f)(x) := \frac{2}{c_{n+1}} \int f\big(x - \xi h(x)\big)\big(1 + |\xi|^2\big)^{-(n+1)/2}\, d\xi \text{ for } x \in \mathbb{R}^n \tag{17}$$

where c_{n+1} is the area of the unit sphere in $\mathbb{R}^{n+1}$. Then

$$T_h : \mathfrak{BC}^{\infty}(\mathbb{R}^n) \longrightarrow \mathfrak{BC}^{\infty}(\mathbb{R}^n) \text{ is linear and continuous} \tag{18}$$

and

$$T_h(f) \text{ is an } \Omega - \text{modification in } \mathfrak{BC}^{\infty}(\mathbb{R}^n) \text{ for any } f \in \mathfrak{BC}^{\infty}(\mathbb{R}^n)\,. \tag{19}$$

T_h is called an Ω - modification operator in $\mathfrak{BC}^{\infty}(\mathbb{R}^n)$. The modification problem was thus reduced to the existence of a function h satisfying (16).

To estimate the derivatives $\partial^a\big(T_h(f)\big)$ for $|a| \leq m$ in Langenbruch [12], we needed the derivatives $\partial^{\beta} f$ for $|\beta| \leq m+1$ if m is odd. So there was a loss of one derivative in these estimates (as in Schmets/Valdivia [17]). This caused a (weak) additional assumption (compared with the papers of Schmets and Valdivia) in [12], when ultradifferentiable functions were considered.

In his Habilitationsschrift [7, chapter 4] Frerick proved that there are extension operators without loss of derivatives. We will show now that a slight variant of the operator T_h from above will give an Ω-modification operator without loss of derivatives (and hence corresponding extension operators with optimal continuity estimates).

More precisely, we will provide an operator T defined and continuous on $\mathfrak{BC}^0(\mathbb{R}^n)$ whose restrictions to $\mathfrak{BC}^m(\mathbb{R}^n)$ are Ω - modification operators in $\mathfrak{BC}^m(\mathbb{R}^n)$ for any $m \in \mathbb{N}_0 \cup \{\infty\}$, that is, they are continuous linear operators within these spaces and $\tilde{f} := T(f)$ is an Ω - modification in $\mathfrak{BC}^m(\mathbb{R}^n)$ for any $f \in \mathfrak{BC}^m(\mathbb{R}^n)$.

For this we need the following interpolation result: For $I :=]-\infty, y]$ and $g \in \mathfrak{BC}^{\infty}(I)$ let

$$|g|_I := \sup_{x \leq y} |g(x)| \text{ and } \|g\|_k := \sup_{a \leq k} |g^{(a)}|_I.$$

Lemma 2 *If $0 \leq g \in \mathfrak{BC}^{\infty}(I)$ is strictly increasing on $[0, y]$ and $g(y) \leq 1$, then*

$$\lim_{y \searrow 0} |g^{(a)}(y)| \ln(1/g(y)) = 0 \textit{ and}$$

$$|g^{(a)}(y)| \ln(1/g(y)) \leq 2^{a+3} \|g\|_{a+1}^{1-2^{-a}} \textit{ for any } a \in \mathbb{N}.$$

PROOF. By a standard interpolation inequality (see Beckenbach, Bellmann [1], p. 171) we have

$$|h'|_I \leq 4|h|_I^{1/2}|h''|_I^{1/2} \text{ if } h \in \mathfrak{BC}^\infty(I).$$

This is applied a times to obtain

$$|g^{(a)}(y)| \leq 4|g^{(a-1)}|_I^{1/2}|g^{(a+1)}|_I^{1/2} \leq 4\ 4^{1/2}|g^{(a-2)}|_I^{1/4}|g^{(a)}|_I^{1/4}\|g\|_{a+1}^{1/2}$$

$$\leq 4^{\sum_{j=0}^{a-1} 2^{-j}}|g|_I^{2^{-a}}\|g\|_{a+1}^{\sum_{j=1}^{a} 2^{-j}} \leq 16\, g(y)^{2^{-a}}\|g\|_{a+1}^{1-2^{-a}}$$

since $0 \leq g$ is increasing. The claim now follows directly since $1 \geq g(y) > 0$ and

$$\sup_{x\in]0,1]} x^\tau \ln(1/x) = \frac{1}{e\tau} \text{ and } \lim_{x\searrow 0} x^\tau \ln(1/x) = 0 \text{ if } \tau \in]0,1[. \quad \blacksquare$$

A modification operator without loss of derivatives is now obtained by means of one extra composition with a function $g \in \mathfrak{BC}^\infty(]-\infty, 2[)$ satisfying the following conditions similar to (16)

$$g\,|_{]0,2[} \text{ is real analytic and strictly increasing, } g\,|_{[0,1]} \leq 1 \text{ and } g\,|_{]-\infty,0]} = 0\,. \tag{20}$$

We finally recall the formula of Fa di Bruno for the derivatives of the composition of two C^∞ - functions v and g of one variable (see e.g. Krantz and Parks [11], Lemma 1.3.1):

$$(v\circ g)^{(b)}(t) = \sum \frac{b!}{k_1!\dots k_b!} v^{(k)}(g(t))\Big(\frac{g^{(1)}(t)}{1!}\Big)^{k_1}\dots\Big(\frac{g^{(b)}(t)}{b!}\Big)^{k_b} \tag{21}$$

where $k := \sum_{j=1}^b k_j$. The sum is taken over all $k_1, \dots k_b$ for which $\sum_{j=1}^b jk_j = b$.

Theorem 4 *Let $h \in \mathfrak{BC}^\infty(\mathbb{R}^n)$ and $g \in \mathfrak{BC}^\infty(]-\infty, 2[)$ satisfy* (16) *and* (20)*, respectively. Let $T(f) := T_{g\circ h}(f)$ be defined for $f \in \mathfrak{BC}^0(\mathbb{R}^n)$ by* (17)*. Then T is an Ω-modification operator in $\mathfrak{BC}^m(\mathbb{R}^n)$ for any $m \in \mathbb{N}_0 \cup \{\infty\}$.*

PROOF. a) T is an Ω - modification operator in $\mathfrak{BC}^0(\mathbb{R}^n)$.
Indeed, let

$$v_f(x,y) := \frac{2}{c_{n+1}} \int \frac{f\big(x-\xi y\big)}{\big(1+|\xi|^2\big)^{(n+1)/2}}\, d\xi \text{ for } (x,y) \in \mathbb{R}^n \times [0,1]$$

and $w_f(x,y) := v_f(x, g(y))$. Thus,

$$T\,(f)\,(x) = w_f\big(x, h\,(x)\big). \tag{22}$$

Since $(1+|\cdot|^2)^{-(n+1)/2} \in L_1(\mathbb{R}^n)$, $T(f)(x)$ is defined for any $x \in \mathbb{R}^n$ and any $f \in \mathfrak{BC}^0(\mathbb{R}^n)$, and

$$\|T(f)\|_{\mathbb{R}^n} \leq \sup_{(x,y)\in\mathbb{R}^n\times[0,1]} |v_f(x,y)| \leq C_1\|f\|_{\mathbb{R}^n} \text{ if } f \in \mathfrak{BC}^0(\mathbb{R}^n). \tag{23}$$

Moreover,

$$w_f \in \mathfrak{BC}^0(\mathbb{R}^n \times [0,1]) \text{ and } w_f(\,\cdot\,,0) = f \text{ if } f \in \mathfrak{BC}^0(\mathbb{R}^n) \tag{24}$$

by (23) and the theorem of dominated convergence. Hence $T(f) \in \mathfrak{BC}^0(\mathbb{R}^n)$ by (22). $T(f)$ is real analytic on Ω since $g\circ h$ is strictly positive and real analytic on Ω and since $v_f(x,y)$ is harmonic, hence real analytic for $y > 0$. Thus, $T(f)$ is an Ω - modification in $\mathfrak{BC}^0(\mathbb{R}^n)$ for any $f \in \mathfrak{BC}^0(\mathbb{R}^n)$ by (24) and (22).

b) To study $T(f)$ for $f \in \mathfrak{BC}^m(\mathbb{R}^n)$, we now estimate $\partial_y v_f$ for $f \in \mathfrak{BC}^1(\mathbb{R}^n)$:

$$|\partial_y v_f(x,y)| \le (C + \ln(1/y)) \max\{\|f\|_{\mathbb{R}^n}, \|\operatorname{grad}(f)\|_{\mathbb{R}^n}\} \text{ if } (x,y) \in \mathbb{R}^n \times\,]0,1].$$

In order to do it, fix $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $\varphi(x) = 1$ if $\|x\| \le 1/2$ and $\varphi(x) = 0$ if $\|x\| \ge 1$. Then

$$\frac{c_{n+1}}{2} v_f(x,y) = \int \frac{f(x-\xi y)\varphi(\xi y)}{(1+|\xi|^2)^{(n+1)/2}} d\xi + \int \frac{f(x-\xi)(1-\varphi(\xi))y}{(y^2+|\xi|^2)^{(n+1)/2}} d\xi,$$

hence

$$\frac{c_{n+1}}{2} \partial_y v_f(x,y) = \int \frac{-\langle \operatorname{grad} f(x-\xi y), \xi\rangle \varphi(\xi y) + f(x-\xi y)\langle \operatorname{grad} \varphi(\xi y), \xi\rangle}{(1+|\xi|^2)^{(n+1)/2}} d\xi$$
$$+ \int f(x-\xi)(1-\varphi(\xi))\Big(\frac{1}{(y^2+|\xi|^2)^{(n+1)/2}} - \frac{(n+1)y^2}{(y^2+|\xi|^2)^{1+(n+1)/2}}\Big) d\xi.$$

Since $\varphi(y\xi) = 0$ if $\|\xi\| \ge 1/y$, the first integral can be estimated by

$$C_1 \max\{\|f\|_{\mathbb{R}^n}, \|\operatorname{grad}(f)\|_{\mathbb{R}^n}\} \int_0^{1/y} \frac{r^{n-1}}{(1+r^2)^{n/2}} dr$$
$$\le C_2 \max\{\|f\|_{\mathbb{R}^n}, \|\operatorname{grad}(f)\|_{\mathbb{R}^n}\} \ln(1+1/y).$$

Similarly, the second integral is estimated by

$$C_3 \|f\|_{\mathbb{R}^n} \int_{\|\xi\| \ge 1/2} (y^2 + |\xi|^2)^{-(n+1)/2} d\xi \le C_4 \|f\|_{\mathbb{R}^n}.$$

c) Let $f \in \mathfrak{BC}^m(\mathbb{R}^n)$ and $m \in \mathbb{N}$. For $j = (a, 2b+l) \in \mathbb{N}_0^{n+1}$, $l = 0,\ 1$ and $|j| \le m$ we have

$$\partial^j v_f(x,y) = (-1)^b \partial_y^l v_{\partial^a \Delta^b f}(x,y) \text{ and } \partial^j w_f(x,y) = \partial_y^{2b+l} w_{\partial^a f}(x,y) \text{ for } y > 0 \tag{25}$$

since $v_f(x,y)$ is harmonic for $y > 0$. By b) we thus get for odd $k \le m$

$$\left|\partial_y^k v_f(x,y)\right| \le (C_5 + \ln(1/y)) C_6^k \sup_{|\alpha| \le k} \left\|\partial^\alpha f\right\|_{\mathbb{R}^n} \text{ if } (x,y) \in \mathbb{R}^n \times\,]0,1]\,. \tag{26}$$

For even $k \le m$, (26) (without the logarithmic term) directly follows from (25) and (23). So (26) holds for any $k \ne 0$. By the formula of Fa di Bruno (21), Lemma 2, (25), (23) and (26) we thus get for $j \in \mathbb{N}_0^{n+1}$ and $0 \le |j| \le m$

$$\left|\partial^j (w_f(x,y))\right| \le C_7 C_8^{|j|} \sup_{|\alpha| \le |j|} \left\|\partial^\alpha f\right\|_{\mathbb{R}^n} \text{ if } (x,y) \in \mathbb{R}^n \times\,]0,\delta]\,, \tag{27}$$

that is, $w_f \in \mathfrak{BC}^m(\mathbb{R}^n \times]0,1])$. Moreover, by the same arguments and (24),

$$\partial^a w_f \in \mathfrak{BC}^m(\mathbb{R}^n \times [0,1]),\ \lim_{y \searrow 0} \partial_x^a w_f(x,y) = \partial^a f(x) \text{ if } |a| \le m, \text{ and}$$

$$\lim_{y \searrow 0} \partial_x^a \partial_y^k w_f(x,y) = 0 \text{ if } k \ne 0 \text{ and } |a| + k \le m. \tag{28}$$

that is, $w_f \in \mathfrak{BC}^m(\mathbb{R}^n \times [0,1])$, $T(f) \in \mathfrak{BC}^m(\mathbb{R}^n)$ and T is a continuous operator in $\mathfrak{BC}^m(\mathbb{R}^n)$.
To prove (15) we notice that $\partial_x^a(T(f)(x)) = \partial_x^a(w_f(x,h(x)))$ consists of the sum of $(\partial_x^a w_f)(x,h(x))$ and certain products each containing $(\partial_x^\alpha \partial_y^k w_f)(x,h(x))$ for some $k \ne 0$ and $|\alpha| + k \le m$. We therefore get by (28)

$$\partial_x^a(T(f)(x)) = (\partial_x^a w_f)(x,0) = \partial^a f(x) \text{ if } |a| \le m \text{ and } x \in F$$

since $h(x) = 0$. The theorem is thus proved. ■

The existence of Ω-modifications in $C^m(\mathbb{R}^n)$ for (unbounded) C^m-functions now easily follows from Theorem 4 and the Approximation Theorem:

Corollary 1 *Let $m \in \mathbb{N}_0 \cup \{\infty\}$. Any $f \in C^m(\mathbb{R}^n)$ has an Ω-modification $\widetilde{f}$ in $C^m(\mathbb{R}^n)$.*

PROOF. By the Approximation Theorem 3b) and c) (for $\Omega = \mathbb{R}^n$) there are $f_1 \in H(\mathbb{C}^n)$ and $f_2 \in \mathfrak{BC}^m(\mathbb{R}^n)$ such that $f = f_1 + f_2$. With the Ω-modification operator T from Theorem 4, we then set $\widetilde{f} := f_1 + T(f_2)$. ■

To consider Ω-modification operators in classes of (M_p)-ultradifferentiable functions, we must choose the function h in (16) such that

$$h \in \mathfrak{BE}_{(L_p)}(\mathbb{R}^n) := \Big\{ f \in C^\infty(\mathbb{R}^n) \;\Big|\; \forall C > 0 :$$
$$\|f\|_{\mathbb{R}^n, C} := \sup\big\{ \big|f^{(\alpha)}(x)\big| / \big(L_{|\alpha|} C^{|a|}\big) \;\big|\; \alpha \in N_0^n, x \in \mathbb{R}^n \big\} < \infty \Big\}$$

where $(L_p)_p$ satisfies $(\mathrm{M3}')$,

$$1_p^* := L_p/(pL_{p-1}) \text{ is increasing} \tag{M1*}$$

and

$$L_p \le C_\varepsilon\, \varepsilon^p\, M_p \text{ for any } \varepsilon > 0\,. \tag{29}$$

(see Langenbruch [12]). Such a function h was constructed in loc. cit. using infinite products and some subtle arguments from the theory of ultradifferentiable functions. We will now show that the existence of h is an easy consequence of the Approximation Theorem 3a):

Proposition 1 *There is $h \in \mathfrak{BE}_{(L_p)}(\mathbb{R}^n)$ satisfying* (16) *if $(L_p)_p$ satisfies* $(\mathrm{M1})$ *and* $(\mathrm{M3}')$.

PROOF. W.l.o.g. let $L_0 = 1$. It is clear that there is $H \in \mathfrak{BE}_{(L_p)}(\mathbb{R}^n)$ satisfying (16) such that $1/4 \ge H > 0$ on Ω. Set $\eta := H/2|_\Omega$ and choose $g \in H_{\mathbb{R}}(\Omega^*)$ by the Approximation Theorem 3a) (and Remark 1) for $f := H|_\Omega$ (and L_p instead of M_p). Then

$$\big|g(x)\big| \le H(x) + \big|g(x) - H(x)\big| \le 1/2 \text{ and}$$
$$\big|g(x)\big| \ge H(x) - \big|g(x) - H(x)\big| \ge H(x)/2 > 0 \text{ if } x \in \Omega\,.$$

Since $H \in \mathfrak{BE}_{(L_p)}(\mathbb{R}^n)$ and $H = 0$ on $F \supset \partial\Omega$, for any $C > 0$ there are $C_1 \ge 1$ and $K \subset\subset \Omega$ such that we get from the Approximation Theorem 3a) for any $a \in \mathbb{N}_0^n$

$$\big|g^{(a)}(x)\big| \le \big|g^{(a)}(x) - H^{(a)}(x)\big| + \big|H^{(a)}(x)\big|$$
$$\le H(x)^{|a|} L_{|a|} + C_1 C^{|a|} L_{|a|} \le 2C_1 C^{|a|} L_{|a|} \text{ if } x \in \Omega \backslash K\,. \tag{30}$$

Since $g \in H(\Omega^*) \subset \mathfrak{E}_{(L_p)}(\Omega)$, this implies that

$$\sup \big\|g^{(a)}\big\|_\Omega C^{|a|} / L_{|a|} < \infty \text{ if } C > 0\,. \tag{31}$$

Set $h(x) := g(x)$ if $x \in \Omega$, and $h(x) = 0$ on F. Then $h \in C^\infty(\mathbb{R}^n)$ and h is flat on F since by (30), $g^{(a)}(x) \longrightarrow 0$ if $x \longrightarrow \partial\Omega$ since H is flat on $F \subset \partial\Omega$. (31) then shows that $h \in \mathfrak{BE}_{(L_p)}(\Omega)$. ■

Since we have no loss of derivatives for the Ω-modification operator in Theorem 4, we get corresponding operators in classes of ultradifferentiable functions without any extra assumption:

Theorem 5 *There is an Ω-modification operator T in $\mathfrak{BE}_{(M_p)}(\mathbb{R}^n)$ if (M_p) satisfies* $(\mathrm{M1})$ *and* $(\mathrm{M3}')$.

PROOF. By Langenbruch [12, Lemma 2.3] we can choose $(L_p)_p$ satisfying $(\mathrm{M3}')$, $(\mathrm{M1}^*)$ and (29). We may assume that $(L_p)_p$ also satisfies

$$L_{p+1} \le A^{p+1} L_p \text{ for any } p \in \mathbb{N}_0. \tag{M2$'$}$$

Choose h for $(L_p)_p$ by Proposition 1. Then choose g_1 for $(L_p)_p$ by Proposition 1 (for $n = 1$, $F :=]-\infty, 0]$ and $\Omega :=]0, \infty[$) and set $g(y) := c\int_{-\infty}^{y} g_1(t)\, dt$. Then g satisfies (20) for suitable c, and the operator $T := T_{g\circ h}$ from Theorem 4 is an Ω - modification operator in $\mathfrak{BC}^\infty(\mathbb{R}^n)$. Using the notation from the proof of Theorem 4, we also know that $w_f \in \mathfrak{BC}^\infty(\mathbb{R}^n \times [0,1])$. By Komatsu [10, Remark after Theorem 4.4], composition with h is continuous in $\mathfrak{E}_{(M_p)}$ since by $(\mathrm{M}3')$

$$M_\gamma(\alpha-\gamma)! \le C^{|\alpha|} M_{|\alpha|} \text{ if } \gamma \le \alpha$$

(Komatsu [9, Lemma 4.1]). We thus have to show that $w_f \in \mathfrak{BE}_{(M_p)}(\mathbb{R}^n \times]0,1])$ if $f \in \mathfrak{BE}_{(M_p)}(\mathbb{R}^n)$. To see this we use the formula of Fa di Bruno (21): Fix $C > 0$. For $b \in \mathbb{N}$ and $y > 0$ we get

$$|\partial_x^a \partial_y^b w_f(x,y)| = |\partial_y^b \big(v_{\partial^a f}(x, g(y))\big)|$$

$$\le \sum \frac{b!}{k_1! \dots k_b!} \big|\big(\partial_y^k v_{\partial^a f}\big)(x, g(y))\big(\frac{g^{(1)}(y)}{1!}\big)^{k_1} \dots \big(\frac{g^{(b)}(y)}{b!}\big)^{k_b}\big|$$

$$\le C_1 2^b C^{|a|} \sum \frac{b!}{k_1! \dots k_b!} (C_2 C)^k M_{|a|+k} \big(\frac{L_2}{1!}\big)^{k_1} \dots \big(\frac{L_{b+1}}{b!}\big)^{k_b}$$

$$\le C_3 (2A)^b C^{|a|} \sum \frac{b!}{k_1! \dots k_b!} (C_4 C)^k M_{|a|+k} \big(L_1^*/(L_0^* l_1^*)\big)^{k_1} \dots \big(L_b^*/(L_0^* l_1^*)\big)^{k_b}$$

$$\le C_5 (2A)^b C^{|a|} \sum \frac{b!}{k_1! \dots k_b!} (C_4 C)^k M_{|a|+k} L_{b-k}^* \le C_5 (4A)^b C^{|a|} \sum \frac{k!}{k_1! \dots k_b!} (C_6 C)^k M_{|a|+k} L_{b-k}$$

$$\le C_7 (4A)^b C^{|a|+b} M_{|a|+b} \sum \frac{k!}{k_1! \dots k_b!} C_6^k \le C_8 (C_9 C)^{|a|+b} M_{|a|+b}$$

where the constants C_k are independent of C, b and y, and where we have used (25), (21), (26), Lemma 2, $(\mathrm{M}2')$ and $(\mathrm{M}1^*)$ for $(L_p)_p$, (29) and finally Lemma 1.3.2 from Krantz and Parks [11]. ■

Notice that $L_p := \prod_{j=1}^{p+1} j\,(\ln(1+j))^2$ satisfies all assumptions needed in the proof of Theorem 5 and that for any $s > 1$

$$L_p \le C_\varepsilon \varepsilon^p (p!)^s \text{ for any } p \in \mathbb{N}.$$

By Theorem 4 and 5, the operator $T = T_{g\circ h}$ constructed with $h, g \in \mathfrak{BE}_{(L_p)}$ is an Ω - modification operator in any of the standard classes of differentiable functions, namely in $\mathfrak{BC}^m(\mathbb{R}^n)$ for $m \in \mathbb{N}_0 \cup \{\infty\}$ and in the Gevrey classes $\gamma^s(\mathbb{R}^n) = \mathfrak{BE}_{(p!^s)}(\mathbb{R}^n)$ and $\Gamma^s(\mathbb{R}^n) = \mathfrak{BE}_{\{p!^s\}}(\mathbb{R}^n)$ for any $s > 1$ since Theorem 5 also holds for the classes $\mathfrak{BE}_{\{M_p\}}(\mathbb{R}^n)$ of ultradifferentiable functions of Roumieu type (by essentially the same proof).

The existence of ultradifferentiable Ω-modifications for unbounded ultradifferentiable functions follows from Theorem 5 as in Corollary 1.

5. A criterion for surjectivity

We will first prove a criterion for surjectivity of continuous linear operators between Fréchet spaces based on the Decomposition Lemma 1. This will be applied to several linear problems of analysis including the representation of ultradistributions as boundary values of holomorphic functions and solvability questions for partial differential operators and convolution operators.

We begin with corresponding examples of decomposition pairs:

Remark 2 Let $E(\Omega)$ be a Fréchet space with the topology defined by an increasing sequence of seminorms $\|\ \|_k$. We assume that

a) $E(\Omega)$ is the space of sections on Ω of a sheaf E on $\mathbb{R}^n$ which has continuous restriction mappings

b) there is $(M_p)_p$ such that

i) $\mathfrak{E}_{(M_p)}(\Omega) \subset E(\Omega)$ is dense and

$$\mathrm{id} : \mathfrak{E}_{(M_p)}(\Omega) \subset E(\Omega) \text{ is continuous}$$

ii) For $\varphi \in D_{(M_p)}(\Omega)$ the mapping

$$M_\varphi : \big(D_{(M_p)}(\Omega), \tau_\Omega\big) \longrightarrow \big(D_{(M_p)}(\Omega), \tau_\Omega\big),\ f \longrightarrow \varphi f\,,$$

is continuous for the topology τ_Ω induced by $E(\Omega)$.

For a (fixed) locally finite resolution of the identity $(\varphi_n)_{n\in\mathbb{N}} \subset D_{(M_p)}(\Omega)$ on Ω and increasing sequences $(j_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ and $(C_n)_{n\in\mathbb{N}} \subset\ [1,\infty[$ we define

$$E\big((j_n),(C_n);\Omega\big) := \Big\{ f \in E(\Omega) \ \Big|\ \forall\, k \in \mathbb{N} : q_k\,(f) := \sum_{n\in\mathbb{N}} \big\| M_{\varphi_n}\,(f)\,\big\|_{j_n+k}\, C_n 2^{nk} < \infty \Big\}\,.$$

Then $\big(E\big((j_n),(C_n);\Omega\big),\ E(\Omega)\big)$ is a decomposition pair.

PROOF. I) $F(\Omega) := E\big((j_n),(C_n);\Omega\big)$ is defined since $M_{\varphi_n}\,(f)$ is defined for $f \in E(\Omega)$ by b)i), b)ii) and continuous extension.

II) Definition 1 i)–iii) is satisfied.
Indeed i) holds by assumption. $D_{(M_p)}(\Omega) \subset F(\Omega)$ by Remark 2b)i) and 2b)ii) since the sum defining $q_k\,(f)$ then is finite. For $f \in F(\Omega)$, the sum $\sum M_{\varphi_n}\,(f)$ absolutely converges in $E(\Omega)$ with limit f (use assumption a)). It is thus clear that $F(\Omega)$ is continuously embedded in $E(\Omega)$. iii) is also evident since the sum defining $q_k\big(M_\varphi\,(f)\big)$ is finite for $\varphi \in D_{(M_p)}(\Omega)$.

III) $F(\Omega)$ is complete. Indeed, if $(f_j)_{j\in\mathbb{N}}$ is a Cauchy sequence in $F(\Omega)$, then by Definition 1ii), $(f_j)_{j\in\mathbb{N}}$ converges in $E(\Omega)$ with limit f.

$$\Big\| M_{\varphi_n}\big(f_j - f\big)\Big\|_{j_n+(k+1)} C_n 2^{n(k+1)} \le \varepsilon + \Big\| M_{\varphi_n}\big(f_j - f_m\big)\Big\|_{j_n+(k+1)} C_n 2^{n(k+1)} \le 2\varepsilon$$

if $j \ge j_0\,(\varepsilon)$ (and $m \ge m_0(\varepsilon, n)$). Thus,

$$q_k(f_j - f) \le \varepsilon \text{ if } j \ge j_0\,,$$

$f \in F(\Omega)$ and $\lim f_j = f$ in $F(\Omega)$. ■

For $E(\Omega)$ as in Remark 2 and a compact $K \subset \Omega$ we set

$$E_0(K) := \big\{ f \in E(\Omega) \ \big|\ \mathrm{supp}\,(f) \subset K \big\} \text{ and } E_0(\Omega) := \bigcup_{K\,\subset\subset\,\Omega} E_0(K)\,.$$

We now prove that continuous linear mappings into $E(\Omega)$ are surjective if the range contains $E_0(\Omega)$ and $H(\Omega^*)$. The latter condition is easily verified in many concrete situations.

Theorem 6 *Let E be a Fréchet space and let $E(\Omega)$ satisfy the assumptions of Remark 2. Let G be a locally convex space containing $E(\Omega)$ as a continuously embedded subspace and let*

$$T : E \longrightarrow G \text{ be linear and continuous.}$$

Then $\mathrm{range}\,(T) \supset E(\Omega)$ *if* $\mathrm{range}\,(T) \supset \big(E_0(\Omega)\,\cup\,H(\Omega^*)\big)$.

PROOF. $E_0(K)$ is a Fréchet space for $K \subset\subset \Omega$ by Remark 2a). The mapping

$$T^{-1} : E_0(K) \longrightarrow E/\ker(T)$$

is defined and closed by assumption. Hence T^{-1} is continuous. Choose a compact exhaustion $(K_n)_{n\in\mathbb{N}}$ of Ω. Then for any $n \in \mathbb{N}$ there are increasing sequences $j(n) \in \mathbb{N}$ and $C(n) \geq 1$ such that for any $g \in E_0(K_n)$ there is $h \in E$ with $T(h) = g$ such that

$$\|h\|_n^E \leq C\,(n)\,\|g\|_{j\,(n)} \text{ if } g \in E_0(K_n)\,. \tag{32}$$

Fix a resolution of the identity (φ_n) as in Remark 2 with $\operatorname{supp}\varphi_n \subset K_n$ and set

$$F(\Omega) := E\big((j\,(n)), \big(C\,(n)\big); \Omega\big)\,.$$

By Remark 2 and the Decomposition Lemma, for $f \in E(\Omega)$ we can then choose $\widetilde{f} \in H(\Omega^*)$ such that $(f - \widetilde{f}) \in F(\Omega)$. Choose $h_n \in E$ for $g_n := M_{\varphi_n}(f - \widetilde{f}) \in E_0(K_n)$ as in (32). Then $h := \sum_n h_n \in E$ exists since

$$\sum_{n\geq k} \|h_n\|_k^E \leq \sum_n \big\|M_{\varphi_n}(f - \widetilde{f})\big\|_{j\,(n)} C\,(n)\,2^n < \infty$$

by (32) and the choice of $F(\Omega)$. Also,

$$T(h) = \sum_n T(h_n) = \sum_n M_{\varphi_n}(f - \widetilde{f}) = f - \widetilde{f}$$

since T is continuous and linear. This proves the theorem since $\widetilde{f} \in H(\Omega^*)$ and thus there is $\widetilde{F} \in E$ with $T(\widetilde{F}) = \widetilde{f}$ by assumption. ■

Theorem 6 can often be used as a substitute for the Mittag-Leffler procedure. Roughly, the approximation procedure is realized in Theorem 6 in the range space of T. No density condition for the kernel spectrum of T is needed.

We now give some standard examples for Remark 2, including Hörmander's spaces $B_{p,k}^{\mathrm{loc}}(\Omega)$, the space $D^m(\Omega)'_b$ of distributions of order m and the spaces of ω-ultradifferentiable functions and ω-ultradistributions (see Björck [3] and Braun, Meise and Taylor [5], which will be used as standard reference).

Here $\omega : [0,\infty[\ \longrightarrow\ [0,\infty[$ is an increasing continuous function such that

$$\omega(2t) \leq A\big(\omega(t) + 1\big) \text{ for any } t \geq 0 \tag{33}$$

$$\int_0^\infty \omega(t)/(1+t^2)\,dt < \infty \tag{34}$$

$$\varphi := \omega \circ \exp \text{ is convex on } \mathbb{R} \text{ and } \lim_{x\longrightarrow\infty} \varphi\,(x)\,/x = \infty\,. \tag{35}$$

(33) - (35) are the conditions $(\alpha) - (\gamma)$ of Braun, Meise and Taylor [5]. By (35) the Young conjugate φ^* of φ is defined. Let

$$\mathfrak{E}_{(\omega)}(\Omega) := \Big\{ f \in C^\infty(\Omega) \;\Big|\; \forall\, K \subset\subset \Omega,\ \forall\, C > 0 :$$

$$|f|_{K,C} := \sup\big\{ \big|f^{(a)}\,(x)\big| \exp\big(-\varphi^*(|a|\,C)/C\big) \big|\ x \in K,\ a \in \mathbb{N}_0^n \big\} < \infty \Big\}$$

$$D_{\{\omega\}}(\Omega) := \big\{ f \in C_0^\infty(\Omega) \;\big|\; \exists\ C > 0 : |f|_{\mathbb{R}^n,C} < \infty \big\}$$

be endowed with their natural Fréchet topology (and (DFS)-topology, respectively).

Example 2 *The following spaces satisfy the assumptions of Remark 2:*

a) i) $\mathfrak{E}_{(M_p)}(\Omega)$

ii) $\mathfrak{E}_{(\omega)}(\Omega)$

b) $C^k(\Omega)$ *for* $k \in \mathbb{N}_0 \cup \{\infty\}$.

c) $L_p^{\mathrm{loc}}(\Omega)$, $1 \leq p < \infty$.

d) $B_{p,k}^{\mathrm{loc}}(\Omega)$, $1 \leq p < \infty$, *for weights* $k \in \mathfrak{K}$ *(see Hörmander [8, section 10.1]).*

e) $D^m(\Omega)'_b := \big(\{f \in C^m(\Omega) \mid \operatorname{supp} f \subset\subset \Omega\}\big)'_b$ *for* $m \in \mathbb{N}_0$.

f) i) $D_{\{M_p\}}(\Omega)'_b$

ii) $D_{\{\omega\}}(\Omega)'_b$.

PROOF. The cases a)i), b), c) and f)i) were already treated in Example 1. For d) we refer to Hörmander [8, Theorems 10.1.7 and 10.1.15]. e) is easy. To prove a)ii) and f)ii), an easy construction provides a sequence (M_p) with $(\mathrm{M1})$ and $(\mathrm{M3}')$ such that $\mathfrak{E}_{(M_p)}(\Omega) \subset \mathfrak{E}_{(\omega)}(\Omega)$. Anyway, the spaces $\mathfrak{E}_{(\omega)}(\Omega)$ could also take the role of $\mathfrak{E}_{(M_p)}(\Omega)$ as general frame in this paper. ■

As a first application of Theorem 6 we consider ultradistributional boundary values of holomorphic functions. The situation is the following:

For $f \in H\big((\mathbb{C}\backslash\mathbb{R})^n\big)$ the boundary value $R(f)$ in the sense of $D_{\{M_p\}}(\mathbb{R}^n)'_b$ is defined if the limit

$$\langle R(f), \varphi \rangle := \sum_{\varepsilon \in \sigma} \lim_{y \longrightarrow 0} \prod_{j=1}^{n} \varepsilon_j \int f(x + i\varepsilon y)\varphi(x)\, dx$$

exists for any $\varphi \in D_{\{M_p\}}(\mathbb{R}^n)$ (here $\sigma := \{1, -1\}^n$). The boundary value problem can now be divided into two parts:

i) Find a weighted space H_W of holomorphic functions defined on $(\mathbb{C}\backslash\mathbb{R})^n$ such that the ultradistributional boundary value $R(f)$ exists for any $f \in H_W$.

ii) Show that any ultradistribution is the boundary value of some $f \in H_W$, i.e. that the boundary value mapping

$$R : H_W \longrightarrow D_{\{M_p\}}(\mathbb{R}^n)' \text{ is surjective}.$$

The second problem is usually solved by means of topological tensorproducts (see Petzsche [15]) while the use of Theorem 6 provides an elementary proof. We need the following assumption: there is $A \in \mathbb{N}$ such that for any $p \in \mathbb{N}$

$$m_p^* + 2 \leq m_{Ap}^* \qquad (\mathrm{M4}')$$

where $m*_p := M_p/\big(M_{p-1}p\big)$. Let

$$M^*(t) := \sup\big\{t^p p! M_0/M_p \mid p \in \mathbb{N}\big\} \text{ for } t \geq 0$$

and let $H\big(\{M_p\}\big)$ denote the set of functions $f \in H\big((\mathbb{C}\backslash\mathbb{R})^n\big)$ such that for any closed cone $\Gamma \subset \{y \in \mathbb{R}^n \mid y_j \neq 0$ for any $j\}$ and any $L > 0$

$$|f(z)| \leq C_L \exp\big(M^*(L/|\operatorname{Im} z|)\big) \text{ if } \operatorname{Im}(z) \in \Gamma.$$

Theorem 7 *Let* (M_p) *satisfy* $(\mathrm{M1}^*)$, $(\mathrm{M2}')$, $(\mathrm{M3}')$, *and* $(\mathrm{M4}')$. *Then*

$$R : H\big(\{M_p\}\big) \longrightarrow D_{\{M_p\}}(\mathbb{R}^n)' \textit{ is surjective}.$$

PROOF. The boundary value $R(f)$ exists for $f \in H\big(\{M_p\}\big)$ by Schroer [20, 5.1] (see also Komatsu [9] and Petzsche/Vogt [16], where slightly stronger assumptions are used). Also by loc. cit., $R\big(H(\{M_p\})\big) \supset \mathfrak{E}_{\{M_p\}}(\mathbb{R}^n)'_b$.

Clearly, $R\big(H(\{M_p\})\big) \supset H(\mathbb{C}^n)$ since

$$R(\widetilde{f}) = f|_{\mathbb{R}^n} \text{ for } f \in H(\mathbb{C}^n)\,,$$

where $\widetilde{f}\,(z) := f\,(z)$ if $\operatorname{Im} z_j > 0$ for any j and $\widetilde{f}\,(z) := 0$ otherwise. The claim now follows from Theorem 6 and Example 2. ■

The boundary value problem for distributions and ultradistributions of Beurling type is much more involved (see Vogt [23] and Petzsche [15]).

As a second application of Theorem 6 we notice that in the situation of Example 2, sections on Ω may be extended modulo $H(\Omega^*)$ to global sections:

Proposition 2 *Let $E(\Omega)$ be one of the spaces from Example 2. Then for any $f \in E(\Omega)$ there are $F \in E(\mathbb{R}^n)$ and $g \in H(\Omega^*)$ such that $f = F|_\Omega + g$.*

PROOF. This follows from Theorem 6 applied to

$$T : E(\mathbb{R}^n) \times H(\Omega^*) \longrightarrow E(\Omega),\, T(F,g) := F|_\Omega + g\,. \quad ■$$

We now consider convolution operators. Let $\mu \in E(\mathbb{R}^n)'$ where

$$E = C^\infty \text{ or } E = \mathfrak{E}_{(\omega)} \text{ or } E = \big(D_{\{\omega\}}\big)'\,. \tag{36}$$

Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open and let $\operatorname{supp}(\mu) + \Omega_1 \subset \Omega_2$. Then the convolution operator

$$\mu * : E(\Omega_2) \longrightarrow E(\Omega_1)$$

is defined, continuous and linear (see e.g. Hörmander [8, chapter 16] and Bonet, Galbis, Meise [2]). We now get the following result:

Theorem 8 *In the above situation the following statements are equivalent:*

a) $\mu * : E(\Omega_2) \longrightarrow E(\Omega_1)$ *is surjective.*

b) i) $\mu * \big(E(\mathbb{R}^n)\big) \supset E_0(\mathbb{R}^n)$ *and*

ii) $\mu * \big(E(\Omega_2)\big) \supset H(\Omega_1^*)$.

PROOF. " b) $\Longrightarrow$ a)" It is well known that

$$\varkappa * : H(\mathbb{C}^n) \longrightarrow H(\mathbb{C}^n) \text{ is surjective} \tag{37}$$

for any $\varkappa \in H(\mathbb{C}^n)'$ (see Ehrenpreis [6] and Malgrange [13]). Hence

$$\mu * \big(E(\mathbb{R}^n)\big) = E(\mathbb{R}^n) \tag{38}$$

by Theorem 6 and b)i). Using (38) and b)ii) Proposition 2 implies that $\mu * \big(E(\Omega_2)\big) = E(\Omega_1)$.

"a) $\Longrightarrow$ b)" ii) clearly holds. i) follows from Hörmander [8, Theorem 16.5.7] and Bonet, Galbis and Meise [2, Proposition 2.6, Corollary 2.9, and Theorem 3.5], respectively. ■

Condition 8 b)i) is equivalent to (38) and to the existence of an elementary solution in $D(\mathbb{R}^n)'$ (and in $D_{(\omega)}(\mathbb{R}^n)'$, and in $D_{\{\omega\}}(\mathbb{R}^n)'$, respectively). Further equivalent conditions for surjectivity of convolution operators are given in the detailed paper of Bonet, Galbis and Meise [2].

Since any partial differential operator $P(D)$ with constant coefficients has a distributional elementary solution, Theorem 8 implies the following result which was first proved by Zampieri [25]:

Corollary 2 *Let $\Omega \subset \mathbb{R}^n$ be open. If $P(D)A(\Omega) = A(\Omega)$, then $P(D)C^\infty(\Omega) = C^\infty(\Omega)$, i.e. Ω is P-convex for supports.* ■

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