Unitary sequences and classes of barrelledness

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Abstract. It is well known that some dense subspaces of a barrelled space could be not barrelled. Here we prove that dense subspaces of $l_\infty(\Omega, X)$ are barrelled (unordered Baire-like or $p-$barrelled) spaces if they have “enough” subspaces with the considered barrelledness property and if the normed space $X$ has this barrelledness property.

These dense subspaces are used in measure theory and its barrelledness is related with some sequences of unitary vectors.

1. Preliminaries

Along this paper $\Omega$ will denote a non void set, $X$ a normed space over the field $K$ of real or complex numbers, $l_\infty(\Omega, X)$ the linear space over $K$ of all those functions $f: \Omega \rightarrow X$ such that the set $\{\|f(\omega)\|: \omega \in \Omega\}$ is bounded, equipped with the supremum norm $\|f\|_\infty = \sup\{\|f(\omega)\|: \omega \in \Omega\}$, $bcs(\Omega, X)$ the linear subspace of $l_\infty(\Omega, X)$ of all those functions $f \in l_\infty(\Omega, X)$ countably supported and $c_0(\Omega, X)$ the linear subspace of $bcs(\Omega, X)$ of all those functions $f: \Omega \rightarrow X$ such that for each $\varepsilon > 0$ the set $\{\omega \in \Omega : \|f(\omega)\| > \varepsilon\}$ is finite or empty.

Let us recall that a (Hausdorff) locally convex space $E$ is barrelled if each barrel (i.e. each absorbing, closed and absolutely convex set) in $E$ is a neighborhood of the origin (see [14, Definition 4.1.1]).

A $p-$net in a vector space $Y$ (see [1]) is a family $\mathcal{W} = \{E_t : t \in T_p\}$ of linear subspaces of $Y$, with $T_p = \bigcup_{k=1}^{p} N^k$, such that $Y = \bigcup_{n \in \mathbb{N}} E_n, E_n \subset E_{n+1}, E_t = \bigcup_{n \in \mathbb{N}} E_{t,n}$ and $E_{t,n} \subset E_{t,n+1}$, for $t \in T_{p-1}$ and $n \in \mathbb{N}$.

A (Hausdorff) locally convex space $E$ is barrelled of class $p$ ($p-$barrelled for short) if given a $p-$net $\mathcal{W} = \{E_t : t \in T_p\}$ there is a $t \in \mathbb{N}^p$ such that $E_t$ is barrelled and dense in $E$. The barrelled spaces of class 1 were introduced by Valdivia in [23] with the name suprabarrelled spaces, also called ($db$)-spaces in [15] and [20].
Other definitions of barrelled spaces related to the Banach-Steinhaus theorem or with the closed graph theorem may be found in [6, Theorems 1.1.4, 1.1.8, 3.2.2 and 3.2.4].

It has been discovered that some of the classical barrelled functional spaces are barrelled spaces of class \( p \). For instance, Dieudonné (cf. [25, p. 133]) proved that \( l_0^{\infty} \) (i.e., the linear subspace of \( l_\infty \) formed by the sequences taking finitely many different values) is barrelled. The barrelledness of \( l_0^{\infty} \) was also pointed out independently by Saxon [19]. If \( A \) is a ring of subsets of \( \Omega \) and \( l_0^{\infty}(A) \) is the linear hull with coefficients in \( K \) of the characteristic functions \( \chi_A, A \in A \), endowed with the supremum norm, Schachermayer [21] noticed that \( l_0^{\infty}(A) \) is barrelled if and only if its dual \( ba(A) \), the vector space over \( K \) of the bounded finitely additive scalar measures defined on \( A \) equipped with the supremum norm, verifies the Nikodým boundedness theorem ([2, p. 80]).

If \( A \) is a \( \sigma \)-algebra Valdivia noticed that \( l_0^{\infty}(A) \) is suprabarrelled [23] and Ferrando and López Pellicer found that \( l_0^{\infty}(A) \) is \( p- \) barrelled [4]. Some other strong barrelledness properties of \( l_0^{\infty}(A) \) and applications may be found in [5], [8], [9], [11], [16] and [17].

It was proved in [13] that if \( \Omega \) is countable infinite then \( c_0(\Omega, X) \) is barrelled if and only if \( X \) is barrelled. For an infinite set \( \Omega \), it has been established in [7] that \( c_0(\Omega, X) \) is barrelled, ultrabornological or unordered Baire-like (22) if and only if \( X \) is barrelled, ultrabornological or unordered Baire-like, respectively. In [12] it has been proved that \( c_0(\Omega, X) \) is 1- barreled if and only if \( X \) is 1- barrelled.

The aim of this paper is to prove that \( c_0(\Omega, X), bcs(\Omega, X) \) and a wide class of subspaces of \( bcs(\Omega, X) \) are (barrelled) \( p- \) barrelled if and only if \( X \) is (barrelled) \( p- \) barrelled.

In what follows \( supp(f) \) means the support of \( f \), i.e. \( supp(f) = \{ x \in \Omega : f(x) \neq 0 \} \). We are going to use the classical notation given, for instance, in [2] and [25]. The linear hull of a subset \( A \) of a linear space \( E \) will be denoted by \( \langle A \rangle \).

If \( E \) is a linear subspace of \( bcs(\Omega, X) \) we will denote by \( S_E \) the family of all sequences \( \{ f_n : n \in \mathbb{N} \} \) such that \( f_n \in E, \| f_n \|_\infty = 1 \) for each \( n = 1, 2, \ldots \) and whose support verify one of the following conditions:

a) \( supp(f_n) \cap supp(f_m) = \emptyset \) if \( n \neq m \)

b) there is a countable set \( \{ w_1, w_2, \ldots, w_n, w_{n+1}, \ldots \} \subset \Omega \) such that \( supp(f_n) \subset \{ w_{n+1}, w_{n+2}, \ldots \} \) for \( n = 1, 2, \ldots \).

If \( f \in E \) and \( \Gamma \subset \Omega \) then \( P_\Gamma f \) is the element of \( bcs(\Omega, X) \) such that \( (P_\Gamma f)(x) = f(x) \) if \( x \in \Gamma \) and \( (P_\Gamma f)(x) = 0 \) when \( x \notin \Gamma \). We will define \( E(\Gamma) = \{ f \in E : supp(f) \subset \Gamma \} \) and, in particular \( bcs(\Gamma, X) = \{ f \in bcs(\Omega, X) : supp(f) \subset \Gamma \} \).

We will denote by \( B \) the family of linear subspaces of \( bcs(\Omega, X) \) such that if \( E \in B \) and \( \Delta \subset \Gamma \subset \Omega \), being \( \Delta \) finite and \( \Gamma \) countable, then \( bcs(\Delta, X) \subset E(\Gamma) = P_\Gamma (E) \). Then \( E = E(\Gamma) + E(\Omega \setminus \Gamma) \).

2. Barrelledness

In the family \( B \) we are going to consider the family \( B_0 \) such that the locally convex vector space \( E \in B \) belongs to \( B_0 \) if given a sequence \( \{ f_n : n \in \mathbb{N} \} \in S_E \) there exists a barrelled space \( (F, \tau) \) such that \( F \subset E \), \( \{ f_n : n \in \mathbb{N} \} \) is bounded in \( (F, \tau) \) and \( \tau \) is a locally convex topology finer than the topology induced in \( F \) by the topology of \( E \).

Lemma 1 If \( E \in B_0 \) and \( Q \) is a barrel in \( E \) there exists a finite set \( \Delta \) (possibly empty) such that \( Q \) absorbs the unit ball of \( E(\Omega \setminus \Delta) \).

Proof. We assert that there is a countable set \( \Delta = \{ w_1, w_2, \ldots \} \) such that \( Q \) absorbs the closed unit ball of \( E(\Omega \setminus \Delta) \). In fact, if this were not true, there would be a \( f_1 \in E \) with \( \| f_1 \|_\infty = 1 \) and \( f_1 \notin Q \). By the hypothesis and the countability of \( \Delta_1 = supp(f_1) \) we deduce the existence of \( f_2 \in E(\Omega \setminus \Delta_1) \) with \( \| f_2 \|_\infty = 1 \) and \( f_2 \notin 2Q \). Once again, as the set \( \Delta_2 = supp(f_2) \) is countable there exists \( f_3 \in E(\Omega \setminus (\{ \Delta_1 \cup \Delta_2 \}) \) with \( \| f_3 \|_\infty = 1 \) and \( f_3 \notin 3Q \).

By induction we would obtain a sequence \( \{ f_n : n \in \mathbb{N} \} \in S_E \). Then, by hypothesis, there exists a barrelled space \( (F, \tau) \) being \( F \subset E \) and \( \tau \) a locally convex topology finer than the topology induced in \( F \)
by the topology of \( E \), such that \( \{ f_n : n \in \mathbb{N} \} \) is bounded in \((F, \tau)\). Therefore \( Q \cap F \) is a 0—neighborhood in \((F, \tau)\) and then, by boundedness, there exists a \( p \) such that \( \{ f_n : n \in \mathbb{N} \} \subset pQ \). From this relation follows the contradiction \( f_p \in pQ \).

Therefore, there exists a countable set \( \Delta = \{ w_1, w_2, \ldots, w_n \} \) such that \( Q \) absorbs the closed unit ball of \( E(\Omega \setminus \Delta) \).

Now we are going to prove that there exists a natural number \( i \) such that \( Q \) absorbs the closed unit ball of \( E(\{ w_{i+1}, w_{i+2}, \ldots \}) \).

If this were not true, there would exist a sequence \( \{ f_n : n \in \mathbb{N} \} \) with \( f_n \in E(\{ w_{n+1}, w_{n+2}, \ldots \}) \), \( \| f_n \|_\infty = 1 \) and \( f_n \notin nQ \) for each \( n = 1, 2, \ldots \). But the sequence \( \{ f_n : n \in \mathbb{N} \} \in S_E \) and then, as in the preceding case, we would obtain a \( q \in \mathbb{N} \) such that \( \{ f_n : n \in \mathbb{N} \} \subset qQ \). This last inclusion contains the contradiction \( f_q \in qQ \), which proves that there exists a natural number \( i \) such that \( Q \) absorbs the closed unit ball of \( E(\{ w_{i+1}, w_{i+2}, \ldots \}) \).

Finally, we have obtained that if \( \Delta = \{ w_1, w_2, \ldots, w_i \} \) then \( Q \) absorbs the closed unit ball of \( E(\Omega \setminus \Delta) = E(\Omega \setminus \{ w_1, w_2, \ldots \}) + E(\{ w_{i+1}, w_{i+2}, \ldots \}) \). □

**Proposition 1** Suppose that \( E \in B_0 \). Then \( E \) is barrelled if and only if \( X \) is barrelled.

**Proof.** If \( Q \) is a barrel in \( E \) then by Lemma 1, there exists a finite set \( \Delta \) such that \( Q \) absorbs the unit ball of \( E(\Omega \setminus \Delta) \). The barrel \( Q \) also absorbs the unit ball of the barrelled space \( E(\Delta) = X^\Delta \) (see [6, Proposition 1.1.13]). From the isomorphism between \( E \) and \( E(\Omega \setminus \Delta) \times E(\Delta) \) it follows that \( Q \) is a neighborhood of zero in \( E \).

Conversely, if \( E \) is barrelled and \( p \in \Omega \) then from the isometry between \( X \) and \( E(\{ p \}) = E / E(\Omega \setminus \{ p \}) \) it follows from [6, Proposition 1.1.9] that \( X \) is barrelled. □

A locally convex space \( E \) is unordered Baire-like ([22]) if given in \( E \) a countable covering \( \{ A_n, n \in \mathbb{N} \} \) of closed absolutely convex subsets of \( E \), there exists an \( A_n \) which is neighbourhood of zero in \( E \).

In the family \( B \) we are going to consider the family \( B_{ab} \) such that the locally convex vector space \( E \in B \) belongs to \( B_{ab} \) if given a sequence \( \{ f_n : n \in \mathbb{N} \} \in S_E \) there exists an unordered Baire-like space \((F, \tau)\) such that \( F \subset E \), \( \{ f_n : n \in \mathbb{N} \} \) is bounded in \((F, \tau)\), and \( \tau \) is a locally convex topology finer than the topology induced in \( F \) by the topology of \( E \).

**Lemma 2** Let \( \mathcal{V} = \{ V_n : n \in \mathbb{N} \} \) be a sequence of absolutely convex and closed subsets of \( E \) such that \( E = \bigcup_{n \in \mathbb{N}} (V_n) \). Suppose that \( E \in B_{ab} \).

Then there exists a subfamily \( \mathcal{W} = \{ W_n : n \in \mathbb{N} \} \) of \( \mathcal{V} \) and a sequence \( \{ \Delta_n : n \in \mathbb{N} \} \) of finite subsets of \( \Omega \) such that, for every \( n \in \mathbb{N} \), \( E(\Omega \setminus \Delta_n) \subset (W_n) \) and
\[
E = \bigcup_{n \in \mathbb{N}} (W_n)
\]

**Proof.** First we are going to prove that there exists \( m \in \mathbb{N} \) and a countable subset \( \Delta_m \) such that \( E(\Omega \setminus \Delta_m) \subset (V_m) \).

In fact, if this were not true we would find a sequence \( \{ f_n : n \in \mathbb{N} \} \) of unitary vectors in \( E(\Omega) \) such that
\[
f_1 \notin (V_1)
\]
and
\[
f_n \in E(\Omega \setminus \bigcup_{i=1}^{n-1} \Delta_i) - (V_n), \quad n = 2, 3, \ldots
\]
where \( \Delta_i = \text{supp}(f_i) \) and \( \| f_i \|_\infty = 1 \) for each \( i \in \mathbb{N} \).

Then, by hypothesis, there exists an unordered Baire-like space \((F, \tau)\) such that \( F \subset E \), \( \{ f_n : n \in \mathbb{N} \} \) is bounded in \((F, \tau)\) and \( \tau \) is a locally convex topology finer than the topology induced in \( F \) by the topology of \( E \). Therefore, there exists a \( V_m \) that contains a neighborhood of zero in \((F, \tau)\), implying that the bounded
set \( \{ f_n : n \in \mathbb{N} \} \) is contained in \( \langle V_m \rangle \). The inclusion \( \{ f_n : n \in \mathbb{N} \} \subset \langle V_m \rangle \) contains the contradiction \( f_m \in \langle V_m \rangle \), proving our first observation.

Therefore, from this property and [22, Theorem 4.1] we have that there exists a subfamily \( \mathcal{W} = \{ W_n : n \in \mathbb{N} \} \) of \( V \) and a sequence \( \{ \Delta_n : n \in \mathbb{N} \} \) of countable subsets of \( \Omega \) such that \( E(\Omega - \Delta_n) \subset \langle W_n \rangle \), for each \( n \in \mathbb{N} \), and \( E = \bigcup_{n \in \mathbb{N}} \langle W_n \rangle \).

From this first property it follows that it is enough to prove the lemma for \( \Omega = \mathbb{N} \).

In this case we are going to prove that there exists some natural number \( m \) such that
\[
E(\mathbb{N} - \{1, 2, \ldots, m \}) \subset \langle V_m \rangle.
\]

If this property were not true we would find a sequence \( \{ f_n : n \in \mathbb{N} \} \) of unitary vectors in \( E(\mathbb{N}) \) such that
\[
f_n \in E(\mathbb{N} - \{1, 2, \ldots, n \}) - \langle V_n \rangle
\]
and we would have that the sequence \( \{ f_n : n \in \mathbb{N} \} \subset \mathcal{S}_E \). By hypothesis, there exists an unordered Baire-like space \((F, \tau)\) such that \( F \subset E \), \( \{ f_n : n \in \mathbb{N} \} \) is bounded in \((F, \tau)\) and \( \tau \) is a locally convex topology finer than the topology induced in \( F \) by the topology of \( E \). Exactly as in the preceding case we would obtain the contradiction \( f_p \in \langle V_p \rangle \), proving the second property we are looking for.

These two properties imply that there exists \( m \in \mathbb{N} \) and a finite subset \( \Delta_m \) such that
\[
E(\Omega - \Delta_m) \subset \langle V_m \rangle
\]
and, then, from [22, Theorem 4.1] it follows the lemma. ■

**Proposition 2** Suppose that \( E \in \mathcal{B}_{ub} \). Then \( E \) is unordered Baire-like if and only if \( X \) is unordered Baire-like.

**Proof.** If \( E \) is unordered Baire-like and \( p \in \Omega \), then from the isometry between \( X \) and \( E(\{p\}) = E/E(\Omega - \{p\}) \) it follows from [6, Proposition 1.3.6] that \( X \) is unordered Baire-like.

Conversely, if \( X \) is unordered Baire-like and \( E \) were not unordered Baire-like, then there exists a sequence \( \{ V_n : n \in \mathbb{N} \} \) of absolutely convex and closed subsets of \( E \) such that
\[
E = \bigcup\{ V_n, n \in \mathbb{N} \}
\]
and each \( V_n \) is not a neighbourhood of zero in the barrelled space \( E \) (see Proposition 1). Then, by barrelledness, we have that
\[
E \not\subset \langle V_n \rangle, \quad n \in \mathbb{N}.
\]

From these relations and Lemma 2 we deduce that there exists a subsequence \( \{ W_n : n \in \mathbb{N} \} \) of \( \{ V_n : n \in \mathbb{N} \} \) and a sequence \( \{ \Delta_n : n \in \mathbb{N} \} \) of finite subsets of \( \Omega \) such that
\[
E = \bigcup_{n \in \mathbb{N}} \langle W_n \rangle
\]
\[
E \not\subset \langle W_n \rangle, \quad n \in \mathbb{N}
\]
\[
E(\Omega - \Delta_n) \subset \langle W_n \rangle, \quad n \in \mathbb{N}
\]

It is clear that we have for each \( n \in \mathbb{N} \) that
\[
E(\Delta_n) \not\subset \langle W_n \rangle
\]
and then there exists for each \( n \) some \( \delta_n \in \Delta_n \) such that
\[
E(\{ \delta_n \}) \not\subset \langle W_n \rangle.
\]
We consider the equivalence relation $R$ in $\mathbb{N}$ defined by the equality $\delta_m = \delta_n$ (i.e. $mRn$ if $\delta_m = \delta_n$). This relation defines a partition $\{F_n, n \in \mathbb{P}\}$ in $\mathbb{N}$, where $\mathbb{P}$ is a finite or countable subset of $\mathbb{N}$.

Let $\{w_n : n \in \mathbb{P}\}$ be the finite or countable subset of $\Omega$ such that $w_n = \delta_s$, being $s$ an arbitrary element of $F_s$. We may rewrite the relations $E(\{\delta_n\}) \not\subset \langle W_n \rangle$, $n \in \mathbb{N}$, in the form:

$$E(\{w_n\}) \not\subset \langle W_n \rangle, \quad m \in F_n, \quad n \in \mathbb{P}.$$ 

We have that the space $X$ is unordered Baire-like and that $E(\{w_n\})$ and $X$ are isometric. Therefore

$$E(\{w_n\}) \not\subset \bigcup_{m \in F_n} \langle W_m \rangle, \quad n \in \mathbb{P}.$$ 

These non-inclusions enable us to choose $f_n \in E(\{w_n\}) - \bigcup_{m \in F_n} \langle W_m \rangle$, $\|f_n\|_\infty = 1$, for each $n \in \mathbb{P}$. Then:

$$\{f_n : n \in \mathbb{P}\} \not\subset \langle W_n \rangle, \quad \forall m \in \mathbb{N}$$

and the hypothesis $E \in \mathcal{B}_{ab}$ implies the existence of an unordered Baire-like space $(F, \tau)$, being $F \subset E$, $\{f_n : n \in \mathbb{P}\}$ bounded in $(F, \tau)$, and the topology $\tau$ is finer than the topology induced by $E$ in $F$. Therefore, there exists some $W_m$ containing a neighbourhood of zero in $(F, \tau)$. This implies the contradiction $\{f_n : n \in \mathbb{P}\} \subset \langle W_n \rangle$ which proves the proposition. □

3. **Barrelledness of class $p$**

Remember that a (Hausdorff) locally convex space $E$ is barrelled of class $p$ (or $p-$barrelled) if given a $p-$net $W = \{E_t : t \in T_p\}$ there is a $t \in \mathbb{N}^p$ such that $E_t$ is barrelled and dense in $E$. It is not difficult to see that when $E$ is $p-$barrelled there are many $E_t$, $t \in \mathbb{N}^p$, which are barrelled and dense in $E$, and the next definitions help us in obtaining the corresponding proof.

**Definition 1** Let $A$ be a subset of the set $\mathbb{N}$ of natural numbers. We will say that $A$ is a set of class 1 (of strict class 1) if $A$ is infinite (if there exists $n_1 \in \mathbb{N}$ such that $A = \{n \in \mathbb{N} : n \geq n_1\}$).

A subset $A$ of $\mathbb{N}^p$ is a set of class $p$ (of strict class $p$) if $A = \bigcup_{b \in B_1} \{b\} \times C_b$, being $B_1$ a set of class $p - 1$ (of strict class $p - 1$) and such that each $C_b$ a set of class 1 (of strict class 1).

It is obvious that a subset $A$ of class $p$ (of strict class $p$) may be written as $A = \bigcup_{b \in B_k} \{b\} \times C_b$, being $B_k$ a set of class $k$ (of strict class $k$) and each $C_b$ a set of class $p - k$ (of strict class $p - k$), with $1 \leq k \leq p - 1$.

Also an easy induction gives us the next result.

**Proposition 3** Let $A$ and $B$ be two sets of $\mathbb{N}^p$, then:

1. If $A$ is a set of strict class $p$ and $B$ is a set of strict class $p$ (of class $p$), then $A \cap B$ is of strict class $p$ (of class $p$).

2. $A$ contains a set of strict class $p$ if and only if $\mathbb{N}^p \setminus A$ does not contain a set of class $p$.

3. If $B$ is a set of class $p$ there exists a bijection $\varphi$ from $B$ onto $\mathbb{N}^p$ such that $\varphi$ preserves the lexicographic order.

The last statement implies that if $W = \{E_t : t \in T_p\}$ is a $p-$net in the $p-$barrelled space $E$ and $B$ is a set of class $p$ then there is a $t \in B$ such that $E_t$ is barrelled and dense in $E$, because the new numeration of $\{E_t : t \in B\}$ with the help of $\varphi$ gives a new $p-$net in $E$. From this observation the next proposition follows easily.
Proposition 4 A (Hausdorff) locally convex space is barrelled of class $p$ if and only if given a $p$-net $W = \{E_t : t \in \mathbb{T}_p\}$ there exists a set $A_p$ of strict class $p$ such that if $t \in A_p$ then $E_t$ is barrelled and dense in $E$.

PROOF. Let $A = \{t \in \mathbb{N}^p : E_t$ is barrelled and dense in $E\}$. From the preceding observation it follows that $B = \mathbb{N}^p \setminus A$ cannot contain a set of class $p$. Then Proposition 3 statement 2, implies that $A$ contains a set of strict class $p$.

If $t = (t_1, t_2, \ldots, t_i, \ldots, t_p) \in \mathbb{T}_p$ and $E_t$ is barrelled and dense in $E$ then, obviously, $E_{t_1 t_2 \ldots t_i}$ is barrelled and dense in $E$, for $1 \leq i \leq p - 1$ ([6, Proposition 1.1.10]).

Recall that a locally convex space $E$ is Baire-like if given an increasing covering $\{A_n : n \in \mathbb{N}\}$ of $E$, being each $A_n$ a closed absolutely convex subset of $E$, there exists an $A_p$ which is a neighborhood of zero ([(18)]). It is obvious that suprabarrelled spaces are Baire-like, that Baire-like spaces are barrelled, that if $\{E_n : n \in \mathbb{N}\}$ is a linear increasing covering of the Baire-like space $E$ there exists an $E_n$ which is dense in $E$ and that if $F$ is barrelled and dense in the Baire space $E$ then $F$ is Baire-like ([6, Propositions 3.1.2 and 3.2.3]). Therefore a locally convex (Hausdorff) space $E$ is $p$–barrelled if given a $p$-net $W = \{E_t : t \in \mathbb{T}_p\}$ in $E$ it is verified one of the following conditions:

1. There exists $t \in \mathbb{N}^p$ such that $E_t$ is barrelled and dense in $E$.
2. There exists $t \in \mathbb{N}^p$ such that $E_t$ is Baire-like.
3. There is a set $A \subset \mathbb{N}^p$ of strict class $p$ such that for each $t \in A$ we have that $E_t$ is barrelled and dense in $E$.
4. There is a set $A \subset \mathbb{N}^p$ of strict class $p$ such that for each $t \in A$ we have that $E_t$ is Baire-like.

In the two last conditions we may omit the word strict.

Now let us suppose that $W = \{F_t : t \in \mathbb{T}_p\}$ is a $p$-net in $E$. Let $T_{n_1 n_2 \ldots n_p}$ be a barrel in $F_{n_1 n_2 \ldots n_p}$, $V_{n_1 n_2 \ldots n_p} = \bigcap_{i=1}^{n_p} Z_{n_1 n_2 \ldots n_p - 2^{i-1}}$, $S_{n_1 n_2 \ldots n_p} = \bigcap_{i=1}^{n_p} Z_{n_1 n_2 \ldots n_p - 2^{i-1}}$, $Z_{n_1 n_2 \ldots n_p} = \bigcap_{i=1}^{n_p} Z_{n_1 n_2 \ldots n_p - 2^{i-1}}$.

It is obvious that if $A$ is a set of strict class $p$ and $F \subset S_{n_1 n_2 \ldots n_p}$ for each $(n_1 n_2 \ldots n_p) \in A$ then we have that $F \subset S_{m_1}$, $F \subset S_{m_2}$, $\ldots$, $F \subset S_{m_1 m_2 \ldots m_p}$ and $F \subset S_{m_1 m_2 \ldots m_p}$ when $(m_1 m_2 \ldots m_p) \in A$.

Lemma 3 Let $W = \{F_t : t \in \mathbb{T}_p\}$ be a $p$-net in $E$ and let $T_{n_1 n_2 \ldots n_p}$ be a barrel in $F_{n_1 n_2 \ldots n_p}$. Suppose that given a sequence $\{f_p : p \in \mathbb{N}\} \subset S_{n_1}$ for each $(n_1, n_2, \ldots, n_p) \in A$. Then there exists a countable set $\Delta$ (possibly empty) and a set $B$ of strict class $p$ such that $E(\Omega \setminus \Delta) \subset S_{n_1 n_2 \ldots n_p}$ for each $(n_1, n_2, \ldots, n_p) \in B$.

PROOF. We are going to prove the lemma by decreasing induction. First we will see that there is a countable set $\Delta$ (possibly empty) and a natural number $n_1$ such that $E(\Omega \setminus \Delta) \subset S_{n_1}$.

In fact, if this were not true then we may find $f_1 \in E$, with $\|f_1\|_\infty = 1$ and $f_1 \notin S_1$. The set $\Delta_1 = supp(f_1)$ is countable and from $E(\Omega \setminus \Delta_1) \not\subset S_2$ we deduce the existence of a $f_2 \in E(\Omega \setminus \Delta_1)$ with $\|f_2\|_\infty = 1$ and $f_2 \notin S_2$. Then put $\Delta_2 = supp(f_2)$ and from $E(\Omega \setminus (\Delta_1 \cup \Delta_2)) \not\subset S_3$ we may suppose that there exists $f_3 \in E(\Omega \setminus (\Delta_1 \cup \Delta_2))$ with $\|f_3\|_\infty = 1$ and $f_3 \notin S_3$. Continuing in this way we determine by induction a unitary sequence $\{f_n : n \in \mathbb{N}\}$ in $E$ and a pairwise disjoint sequence $\{\Delta_n : n \in \mathbb{N}\}$ of countable subsets of $\Omega$ such that $\Delta_n = supp(f_n)$, $\|f_n\|_\infty = 1$ and $f_n \notin S_n$, for $n = 1, 2, \ldots$.

The sequence $\{f_q : q \in \mathbb{N}\} \subset S_E$. By hypothesis there exists a set $A$ of strict class $p$ such that $\{f_q : q \in \mathbb{N}\} \subset S_{n_1 n_2 \ldots n_p}$, for each $(n_1, n_2, \ldots, n_p) \in A$. We also have that $\{f_q : q \in \mathbb{N}\} \subset S_{n_1}$ if

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(n_1, n_2, \ldots, n_p) \in A$. From this inclusion follows the contradiction $f_{n_1} \in S_{n_1}$, which proves our assertion.

Continuing with the induction let us suppose that there exists a set $A_{h-1}$ of strict class $h-1$ and a countable set $\Delta$ such that $E(\Omega \setminus \Delta) \subset S_{n_1n_2\ldots n_h}$ for each $(n_1, n_2, \ldots, n_{h-1}) \in A_{h-1}$. Let $A'_{h-1}$ and $A''_{h-1}$ be a partition of $A_{h-1}$ such that:

- $A'_{h-1}$ is formed by the elements $b = (n_1, n_2, n_3, \ldots, n_{h-1})$ belonging to $A_{h-1}$ for which we could determine a countable set $\Delta_b$ of $\Omega$ and a natural number $m$ such that $E(\Omega \setminus \{\Delta \cup \Delta_b\}) \subset S_{n_1n_2\ldots n_{h-1}m}$.

- $A''_{h-1} = A_{h-1} \setminus A'_{h-1}$. If $(n_1, n_2, n_3, \ldots, n_{h-1}) \in A_{h-1}$, $\Delta'$ is a countable subset of $\Omega$ and $m \in \mathbb{N}$ we have that $E(\Omega \setminus \{\Delta \cup \Delta'\}) \not\subset S_{n_1n_2\ldots n_{h-1}m}$.

If $A'_{h-1}$ contains a set $B$ of strict class $h-1$ then we immediately obtain the next step of the inductive process. In fact, if $\Delta' = \bigcup_{b \in B} \Delta_b$ we have that for every $b \in B$ there exists a set $I_b$ of strict class 1 such that $E(\Omega \setminus \{\Delta \cup \Delta'\}) \subset S_{n_1n_2\ldots n_{h-1}n_h}$ for every $(n_1, n_2, \ldots, n_{h-1}, n_h) \in \bigcup_{b \in B} \{b\} \times I_b = A_h$, being obvious that $A_h$ is a set of strict class $h$.

If $A'_{h-1}$ does not contain a set of strict class $h-1$ then $A''_{h-1}$ contains a set $B_{h-1}$ of strict class $h-1$ such that for each $(n_1, n_2, \ldots, n_{h-1}, n_h) \in B_{h-1} \times \mathbb{N}$ and each countable subset $\Delta''$ of $\Omega$ we have that

$$E(\Omega \setminus \{\Delta \cup \Delta''\}) \not\subset S_{n_1n_2\ldots n_{h-1}n_h} \quad (1)$$

It is obvious that $B_{h-1} \times \mathbb{N}$ is a set of class $h$, whose elements can be enumerated in the following way $$\{(n_1(i), n_2(i), \ldots, n_{h-1}(i), n_h(i)) : i = 1, 2, 3, \ldots\}.$$ From (1) we deduce that $E(\Omega \setminus \Delta) \not\subset S_{n_1(1), n_2(1), \ldots, n_h(1)}$ which enables us to determine $g_1 \in E(\Omega \setminus \Delta)$, with $\|g_1\|_\infty = 1$ and $g_1 \notin S_{n_1(1), n_2(1), \ldots, n_h(1)}$.

If $\Delta'_1 = \text{supp}(g_1)$ we have by (1) that $E(\Omega \setminus \{\Delta \cup \Delta'_1\}) \not\subset S_{n_1(2), n_2(2), \ldots, n_h(2)}$. This relation indicates the existence of $g_2 \in E(\Omega \setminus \{\Delta \cup \Delta'_1\})$, with $\|g_2\|_\infty = 1$ and $g_2 \notin S_{n_1(2), n_2(2), \ldots, n_h(2)}$.

Now, if $\Delta'_2 = \text{supp}(g_2)$, we also have by (1) that $E(\Omega \setminus \{\Delta \cup \Delta'_1 \cup \Delta'_2\}) \not\subset S_{n_1(3), n_2(3), \ldots, n_h(3)}$.

Therefore, and after an obvious induction, we could obtain a sequence $\{g_i : i \in \mathbb{N}\}$ of unitary vectors in $E$ with pairwise disjoint supports $\Delta'_i = \text{supp}(g_i)$, $i = 1, 2, 3, \ldots$, such that $g_i \notin S_{n_1(i), n_2(i), \ldots, n_h(i)}$.

The sequence $\{g_i : i \in \mathbb{N}\} \subset S_{n_1n_2\ldots n_h}$ and therefore there exists a set $C$ of strict class $p$ such that $\{g_i : i \in \mathbb{N}\} \subset S_{n_1n_2\ldots n_h}$ for $(n_1, n_2, \ldots, n_h, n_{h+1}, \ldots, n_p) \in C$. By Proposition 3 statement 1 there exists an index $k$ such that $\{g_i : i \in \mathbb{N}\} \subset S_{n_1(k), n_2(k), \ldots, n_h(k)}$.

This relation contains the contradiction $g_k \in S_{n_1(k), n_2(k), \ldots, n_h(k)}$ that proves this lemma.

\textbf{Lemma 4} Let $W = \{F_t : t \in T_p\}$ be a $p$-net in the (Hausdorff) locally convex space $E$ and let $T_{n_1n_2\ldots n_p}$ be a barrel in $F_{n_1n_2\ldots n_p}$. If $F$ is a $p$-barrelled subspace of $E$ then there exists a subset $A$ of strict class $p$ such that $F \subset S_{n_1n_2\ldots n_p}$ whenever $(n_1n_2\ldots n_p) \in A$.

\textbf{Proof.} Since $\{F \cap F_t : t \in T_p\}$ is a $p$-net in $F$, there is a set $A$ of strict class $p$ such that if $(n_1n_2\ldots n_p) \in A$ then $F \cap F_{n_1n_2\ldots n_p}$ is barrelled and dense in $F$. By density, $F \cap T_{n_1n_2\ldots n_p}$ is a neighborhood of zero in $F$. Therefore $F \subset S_{n_1n_2\ldots n_p}$ for every $(n_1n_2\ldots n_p) \in A$, which implies that $F \subset S_{n_1n_2\ldots n_p}$ whenever $(n_1n_2\ldots n_p) \in A$.

\textbf{Lemma 5} Let $W = \{F_t : t \in T_p\}$ be a $p$-net in the (Hausdorff) locally convex space $E$, let $T_{n_1n_2\ldots n_p}$ be a barrel in $F_{n_1n_2\ldots n_p}$, $F$ a subspace of $E$ and $\tau$ a locally convex topology in $F$ finer than the induced by $E$. If $(F, \tau)$ is $p$-barrelled then there exists a set $A$ of strict class $p$ such that $F \subset S_{n_1n_2\ldots n_p}$ whenever $(n_1n_2\ldots n_p) \in A$.

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Proof. Since \((F, \tau)\) is \(p\)-barrelled and \(\{F \cap T : t \in T_p\}\) is a \(p\)-net in \(F\), there is a set \(A\) of strict class \(p\) such that if \((n_1, n_2, \ldots, n_p) \in A\) then \(F \cap T_{n_1, n_2, \ldots, n_p}\) is barrelled and dense in \((F, \tau)\). Hence \(F \cap T_{n_1, n_2, \ldots, n_p}\) is a neighbourhood of zero in \(F \cap T_{n_1, n_2, \ldots, n_p}\) endowed with the topology induced by \(\tau\) and, by density, \(F \cap T_{n_1, n_2, \ldots, n_p}\) is a neighbourhood of zero in \((F, \tau)\). From \(F \cap T_{n_1, n_2, \ldots, n_p} \subset T_{n_1, n_2, \ldots, n_p}^E\) it follows that \(F \subset Z_{n_1, n_2, \ldots, n_p}\) for every \((n_1, n_2, \ldots, n_p) \in A\), which implies that \(F \subset S_{n_1, n_2, \ldots, n_p}\) whenever \((n_1, n_2, \ldots, n_p) \in A\). \(\blacksquare\)

Now, in the family \(B\) we are going to consider the subfamily \(B_p\) such that \(E \in B\) belongs to \(B_p\), if given the sequence \(\{f_n : n \in \mathbb{N}\} \subset S_E\) there exists a \(p\)-barrelled space \((F, \tau)\) such that \(\{f_n : n \in \mathbb{N}\} \subset F \subset E\) and \(\tau\) is a locally convex topology finer than the topology induced in \(F\) by the topology of \(E\).

**Lemma 6** Let \(W = \{F : t \in T_p\}\) be a \(p\)-net in \(E\) and let \(T_{n_1, n_2, \ldots, n_p}\) be a barrel in \(F_{n_1, n_2, \ldots, n_p}\). If \(E \in B_p\) and \(X\) is a \(p\)-barrelled space then there exists a set \(A\) of strict class \(p\) such that \(E = S_{n_1, n_2, \ldots, n_p}\) when \((n_1, n_2, \ldots, n_p) \in A\).

Proof. As \(E \in B_p\), Lemma 5 guaranties that we may apply Lemma 3. Therefore it is enough to prove this lemma where \(\Omega = \mathbb{N}\) is a countable set. Therefore we are going to suppose that \(\Omega = \mathbb{N}\) and we will obtain the proof by decreasing induction.

Now we are going to prove that there exists a natural number \(n_1\) such that \(E = S_{n_1}\). In first place, we will find a natural number \(i\) such that \(E(\{i + 1, i + 2, \ldots\}) \subset S_i\). In fact if this were not possible we would determine a sequence \(\{f_i : i \in \mathbb{N}\}\) of unitary vectors such that \(f_i \in E(\{i + 1, i + 2, \ldots\}) \setminus S_i\) for \(i = 1, 2, \ldots\)

The relation \(\{f_i : i \in \mathbb{N}\} \subset S_E, E \in B\) and Lemma 5 implies that there exists a set \(A\) of strict class \(p\) such that \(\{f_i : i \in \mathbb{N}\} \subset S_{n_1, n_2, \ldots, n_p}\) when \((n_1, n_2, \ldots, n_p) \in A\). Therefore if \((n_1, n_2, \ldots, n_p) \in A\) we have that \(\{f_i : i \in \mathbb{N}\} \subset S_{n_1}\) implying the contradiction \(f_{n_1} \in S_{n_1}\).

This enables us to suppose that there exists a natural number \(i\) such that \(E(\{i + 1, i + 2, \ldots\}) \subset S_i\). Let \(\Delta = \{1, 2, \ldots, i\}\).

Since the space \(E(\Delta)\) is isometric to \(X^\Delta\) endowed with the \(l_\infty\) norm, we have that \(E(\{1, 2, \ldots, i\})\) is \(p\)-barrelled (see [6, Proposition 2.3]). Hence, there is a set \(B\) of strict class \(p\) such that \(X^\Delta \subset S_{n_1, n_2, \ldots, n_p}\) when \((n_1, n_2, \ldots, n_p) \in B\) (Lemma 4). Then for each \((m_1, m_2, \ldots, m_p) \in B\) we have that \(X^\Delta \subset S_{m_1}\). If \(n_1 \geq \max(i, m_1)\) then \(E = E(\{1, 2, \ldots, i\}) + E(\{i + 1, i + 2, \ldots\}) \subset S_i + S_{m_1} \subset S_{n_1} + S_{n_1} \subset S_{n_1}\).

Let us suppose that in the \((h - 1)\)-step of the inductive process we have determined a set \(A_{h-1}\) of class \(h - 1\) such that \(E(\mathbb{N}) = S_{n_1, n_2, \ldots, n_{h-1}}\) whenever \((n_1, n_2, \ldots, n_{h-1}) \in A_{h-1}\). Let \(b = (n_1, n_2, \ldots, n_{h-1}) \in A_{h-1}\). The sets \(S'_{n_1, n_2, \ldots, n_h}\), with \(n_i \in \mathbb{N}, h \leq i \leq s \leq p\) generate in a natural way the \(p - (h - 1)\)-net formed by the sets \(S'_{n_1, n_2, \ldots, n_h}\) with \(n_i \in \mathbb{N}\) for \(h \leq i \leq s \leq p\), given by

\[
S'_{n_1, n_2, \ldots, n_h} = S_{n_1, n_2, \ldots, n_h}
\]

and

\[
S'_{n_1, n_2, \ldots, n_{h-1}, n_h} = S_{n_1, n_2, \ldots, n_{h-1}, n_h} \cap S'_{n_1, n_2, \ldots, n_{h-1}, n_h - 1}
\]

with \(n_i \in \mathbb{N}, h \leq i \leq s \leq p\) and \(h \leq i \leq s - 1\).

In \(S'_{n_1, n_2, \ldots, n_p}\) consider the barrel \(T'_{n_1, n_2, \ldots, n_p} = T_{n_1, n_2, \ldots, n_p}^E(\mathbb{N}) \cap S'_{n_1, n_2, \ldots, n_p}\), being \(T_{n_1, n_2, \ldots, n_p}\) the barrel given in \(F_{n_1, n_2, \ldots, n_p}\) \((\subset S'_{n_1, n_2, \ldots, n_p} \subset S_{n_1, n_2, \ldots, n_p})\). Since \(T_{n_1, n_2, \ldots, n_p}\), being the first step of the inductive process to the \(p - (h - 1)\)-net

\[
\{S'_{n_1, n_2, \ldots, n_{h-1}, n_h} : n_i \in \mathbb{N} \text{ for } h \leq i \leq s \leq p\}
\]

we get a set \(I_{b}\) of class \(1\) such that \(E(\mathbb{N}) = S_{n_1, n_2, \ldots, n_{h-1}, n_h}\) for each \(n_h \in I_{b}\). The induction finishes with the observation that \(\bigcup_{b \in A_{h-1}} \{b\} \times I_b\) is a set \(A_h\) of strict class \(h\) and that \(E(\mathbb{N}) = S_{n_1, n_2, \ldots, n_{h-1}, n_h}\) when \((n_1, n_2, \ldots, n_h) \in A_h\). This induction proves the lemma. \(\blacksquare\)
Theorem 1 Let $\Omega$ be a non void set and suppose that $E \in \mathcal{B}_p$. The locally convex space $E$ is $p$-barrelled if and only if $X$ is $p$-barrelled.

PROOF. For $\omega \in \Omega$ we have that $E$ and the product $E \left(\Omega \setminus \{\omega\}\right) \times E \left(\{\omega\}\right)$ with the supremum norm are isometric. Therefore, the spaces $E \left(\{\omega\}\right)$, and $E/E(\Omega \setminus \{\omega\})$ are isometric. Hence if $E$ is $p$-barrelled, then $X$ (isometric to $E \left(\{\omega\}\right)$) is $p$-barrelled according to [6, Proposition 3.2.12].

Conversely, if $X$ is $p$-barrelled we are going to show by induction that $E$ is $p$-barrelled.

Now we prove that $E$ is suprabarrelled (1-barrelled). We know that $E$ is Baire-like (see Proposition 1 and [6, Proposition 1.2.1]). Therefore if $\{F_n : n \in \mathbb{N}\}$ is a 1-net in $E$ there is a $n$ such that $F_n$ is dense in $E$ for $m \geq n$. If $E$ were not suprabarrelled we would find in $E$ a 1-net $\{F_n : n \in \mathbb{N}\}$ such that each $F_n$ is dense and non barrelled. Let $T_n$ be a barrel in $F_n$ which is not neighborhood of zero in $F_n$. Set $V_n = T_n^c$ and $S_n = \bigcap_{m \geq n} (V_n)$. According to Lemma 6, there is some $S_n = E$, hence $E = \{V_n\}$ and the barrelledness of $E$ (Proposition 1) yields that $V_n$ is a neighborhood of zero in $E$. Then $T_n = V_n \cap F_n$ is a neighborhood of zero in $F_n$, a contradiction that enables us to establish that there exists a $F_n$ which is barrelled.

Assuming that $E$ is $(p-1)$-barrelled, $p \geq 2$, and that $\{F_t : t \in T_p\}$ is a $p$-net in $E$, then we can suppose that there is a set $A_{p-1}$ of class $(p-1)$ such that $F_t$ is barrelled and dense in $E$ for $t \in A_{p-1}$. If $t \in A_{p-1}$ then $F_t$ is Baire-like ([6, Proposition 1.2.1]), hence there is a set $A_p$ of class $p$ such that $F_{(n_p)}$ is dense in $E$ for $(t, n_p) \in A_p$. Consequently, if $E$ were not $p$-barrelled we may find a $p$-net $\{F_t : t \in T_p\}$ such that each $F_t$, for $t \in \mathbb{N}^p$, is not barrelled and dense in $E$. Let $T_t$ be a barrel in $F_t$ for $t \in \mathbb{N}^p$, which is not neighborhood of zero in $F_t$, for $t \in \mathbb{N}^p$. According to Lemma 6, there is a $(n_1, n_2, \ldots, n_p)$ such that $S_{n_1 n_2 \ldots n_p} = E$ and then, by barrelledness (Proposition 1), $V_{n_1 n_2 \ldots n_p} = \bigcap_{t \in \mathbb{N}^p} F_t$ is a neighborhood of zero in $S_{n_1 n_2 \ldots n_p}$. This implies the contradiction that $T_{n_1 n_2 \ldots n_p} = V_{n_1 n_2 \ldots n_p} \cap F_{n_1 n_2 \ldots n_p}$ is a zero neighborhood in $F_{n_1 n_2 \ldots n_p}$. Hence $E$ is $p$-barrelled. ■

If we apply Lemma 6 when $T_{n_1 n_2 \ldots n_p} = F_{n_1 n_2 \ldots n_p}$ we obtain the following property.

Let $W = \{F_t : t \in T_p\}$ be a $p$-net in $E$. If $E \in \mathcal{B}_p$ and $X$ is a $p$-barrelled space, then there exists a set $A$ of strict class $p$ such that $F_t$ is dense in $E$ when $t \in A$.

This property simplifies the second part of the proof of Theorem 1. In fact, if $X$ is $p$-barrelled and $E$ were not $p$-barrelled we may find a $p$-net $\{F_t : t \in T_p\}$ such that $F_t$ is not barrelled and dense in $E$, for each $t \in \mathbb{N}^p$. We obtain the same contradiction as in the end of the proof of Theorem 1.

4. Notes

When $E$ is $c_0(\Omega, X)$ or bcs$(\Omega, X)$ and $\{f_n : n \in \mathbb{N}\}$ is a sequence of unitary vectors with disjoint supports, it is easy to prove that $\{\sum_{n=1}^{\infty} \alpha_n f_n : |\alpha_n| \leq 1, n = 1, 2, \ldots\}$ is a Banach disk.

If $\{f_n : n \in \mathbb{N}\}$ is a sequence of unitary vectors of $E$ and there is a countable set $\{w_1, w_2, \ldots\} \subset \Omega$, such that $\text{supp}(f_n) \subset \{w_{n+1}, w_{n+2}, \ldots\}$ then it is obvious that $\{\sum_{n=1}^{\infty} \alpha_n f_n : \sum_{n=1}^{\infty} |\alpha_n| \leq 1\}$ is a Banach disk.

Therefore, $c_0(\Omega, X)$ and bcs$(\Omega, X)$ are $p$-barrelled if and only if $X$ is $p$-barrelled.

References


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