Explosive solutions of semilinear elliptic systems with gradient term

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Abstract. We study the existence of boundary blow-up solutions to the nonlinear elliptic system

\[ \Delta u + |\nabla u| = p(|x|)f(v), \quad \Delta v + |\nabla v| = q(|x|)g(u) \]

in \( \Omega \). Here \( \Omega \) is either a bounded domain in \( \mathbb{R}^N \) or it denotes the whole space. The nonlinearities \( f \) and \( g \) are positive and continuous, while the nonnegative potentials \( p \) and \( q \) are continuous and satisfy appropriate growth conditions at infinity. We show that boundary blow-up positive solutions fail to exist if \( f \) and \( g \) are sublinear. This result holds both if \( \Omega \) is bounded, and if \( \Omega \) is the whole space but \( p \) and \( q \) have slow decay at infinity. We establish the existence of infinitely many entire blow-up solutions in the case where \( p \) and \( q \) are of fast decay and if \( f \) and \( g \) satisfy a sublinear type growth condition at infinity.

1. Introduction and the main results

Existence and nonexistence of solutions of the semilinear elliptic system

\[ \begin{align*}
\Delta u &= f(x, u, v) \quad \text{in } \Omega, \\
\Delta v &= g(x, u, v) \quad \text{in } \Omega
\end{align*} \quad (1) \]

have received much attention recently. See, for example, Chen and Lu [2], Cîrstea and Rădulescu [4], Clément, Manásevich and Mitidieri [5], Dalmasso [6], De Figueiredo and Jianfu [7], Lair and Shaker [14], Serrin and Zou [18, 19], Yarur [20], Wang and Wood [21], and the references therein. Most of these results have to do with the nonexistence of positive solutions, the existence of radial solutions, or the asymptotic behavior of solutions.

Palabras clave / Keywords: semilinear elliptic system, explosive solution, existence and nonexistence results, multiplicity of solutions.

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We are concerned in this paper with the study of positive solutions to the following class of semilinear elliptic systems with gradient term

\[
\begin{align*}
\Delta u + |\nabla u| &= p(|x|) f(v) \quad \text{in } \Omega, \\
\Delta v + |\nabla v| &= q(|x|) g(u) \quad \text{in } \Omega,
\end{align*}
\]

(2)

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) denotes either a bounded open set in $\mathbb{R}^N$ or the whole of $\mathbb{R}^N$. Throughout this paper we assume that \( p, q \not\equiv 0 \) are nonnegative Hölder functions. We also assume that \( f \) and \( g \) are Hölder, positive and non-decreasing functions on \((0, \infty)\).

We are mainly interested in finding properties of large (explosive, blow-up) solutions of (2), that is, positive solutions \((u, v)\) satisfying \( u(x) \to +\infty \) and \( v(x) \to +\infty \) as \( \text{dist} (x, \partial \Omega) \to 0 \) (if \( \Omega \) is bounded), or \( u(x) \to +\infty \) and \( v(x) \to +\infty \) as \( |x| \to \infty \) (if \( \Omega = \mathbb{R}^N \)).

A geometric motivation in that sense can be found in [3, 12, 15]. We also point out the pioneering work of Keller [10] and Osserman [16].

The corresponding equation that leads us to the system (2) is

\[ \Delta u + |\nabla u|^a = p(x) f(u) , \quad x \in \Omega, \quad 0 < a \leq 2, \]

which was treated in [1, 8] (in the case where \( \Omega \) is bounded) and in [9, 13] (for \( \Omega = \mathbb{R}^N \)). Problems of this type arise in stochastic control theory and have been first studied in Lasry and Lions [11]. The corresponding parabolic equation was considered in Quiti\-\-\-nner [17]. In terms of the dynamic programming approach, an explosive solution of (2) corresponds to a value function (or Bellman function) associated to an infinite exit cost (see [11]).

Our first result asserts that if \( \Omega \) is bounded and if both \( f \) and \( g \) are sublinear at infinity, then problem (2) has no positive boundary blow-up solution. More precisely, the following hold

**Theorem 1** Suppose \( \Omega \subset \mathbb{R}^N \) is a bounded domain and \( f, g \) satisfy

\[ \max \left\{ \sup_{t \geq 1} f \left( \frac{t}{t} \right), \sup_{t \geq 1} g \left( \frac{t}{t} \right) \right\} < +\infty. \]

then problem (2) has no positive large solution.

The same conclusion holds if \( \Omega = \mathbb{R}^N \), but under natural additional assumptions related to the behavior of \( p \) and \( q \) at infinity. In order to state the result in this case, let us first define, for any \( r \geq 0 \),

\[ P(r) = \int_0^r e^{r t^{N-1}} p(t) dt, \quad Q(r) = \int_0^r e^{r t^{N-1}} q(t) dt. \]

(3)

**Theorem 2** Let \( \Omega = \mathbb{R}^N \). Assume that \((A_1)\) holds and

\[ \int_1^{\infty} P(r) dr < +\infty, \quad \int_1^{\infty} Q(r) dr < +\infty. \]

then problem (2) has no positive entire large solution.

**Theorem 3** Let \( \Omega = \mathbb{R}^N \). Assume that

\[ \int_1^{\infty} P(r) dr = +\infty, \quad \int_1^{\infty} Q(r) dr = +\infty. \]

If

\[ \lim_{t \to \infty} \frac{f(a g(t))}{t} = 0, \quad \text{for all constants } a \geq 1, \]

(4)

then problem (2) has infinitely many positive entire large solutions.
We point out that Condition \((A_2)\) has been introduced in [4].

\textbf{Remark 1} Using the fact that
\[
\int_0^r e^r t^k dt = k! e^r \sum_{s=1}^k (-1)^{k-s} \frac{t^s}{s!} \quad \text{for all integers } k \geq 1, \tag{6}
\]
we observe that the following functions verify (4) or (5):

(i) condition (4) holds provided that \(p(t) = \frac{1}{1 + t^\gamma}, \gamma > 1\) and \(q(t) = \frac{1}{(1 + t^2)^\theta}, \theta > \frac{1}{2}\).

(ii) condition (5) holds provided that \(p(t) = t^\gamma, q(t) = t^\theta, \gamma, \theta \geq 0\). \(\blacksquare\)

\textbf{Remark 2} We give in what follows some examples of nonlinearities \(f\) and \(g\) that satisfy \((A_2)\):

(i) \(f(t) = \sum_{j=1}^n a_j t^{\gamma_j}, g(t) = \sum_{k=1}^m b_k t^{\theta_k}, t \geq 0\) with \(a_j, b_k, \gamma_j, \theta_k > 0\) and \(\gamma \theta < 1\), where \(\gamma = \max_{1 \leq j \leq n} \gamma_j, \theta = \max_{1 \leq k \leq m} \theta_k\).

(ii) \(f(t) = (1 + t^\gamma)^{\gamma_2}, g(t) = (1 + t^\theta)^{\theta_2}, \gamma_1, \gamma_2, \theta_1, \theta_2 > 0\) and \(\gamma_1 \gamma_2 \theta_1 \theta_2 < 1\).

(iii) \(f(t) = \ln(1 + t^\gamma), g(t) = \ln(1 + t^\theta), \gamma, \theta > 0\).

(iv) \(f(t) = \ln(1 + t^\gamma), g(t) = e^{t^\theta}, \gamma > 0, \theta \in (0, 1)\). \(\blacksquare\)

\section{Proof of Theorem 1}

Suppose that \((u, v)\) is a positive large solution of \((2)\) and let \(w(x) = \ln(1 + u(x) + v(x)), x \in \Omega\). Then \(w\) is a positive function and \(w(x) \to \infty\) as \(\text{dist}(x, \partial \Omega) \to 0\). A simple calculation yields

\[
\Delta w = \frac{\Delta u + \Delta v}{1 + u + v} - \sum_{i=1}^N (u_{x_i} + v_{x_i})^2 \quad \text{in } \Omega.
\]

Taking into account the assumption \((A_1)\) we have

\[
\Delta w \leq \frac{\Delta u + \Delta v}{1 + u + v} \leq \frac{\|p\|_{L^\infty(\Omega)} f(v) + \|q\|_{L^\infty(\Omega)} g(u)}{1 + u + v} \leq \left(\frac{\|p\|_{L^\infty(\Omega)} + \|q\|_{L^\infty(\Omega)}}{1 + u + v} \right) \left(\frac{f(v) + g(u)}{1 + u + v} \right) \leq K,
\]

for some constant \(K > 0\). Hence

\[
\Delta(w(x) - K|x|^2) < 0, \quad \text{for all } x \in \Omega.
\]

Let \(z(x) = w(x) - K|x|^2, x \in \Omega\). Then

\[
\Delta z < 0 \quad \text{in } \Omega \tag{7}
\]

and

\[
(z(x) \to \infty \quad \text{as } \text{dist}(x, \partial \Omega) \to 0. \tag{8}
\]

Fix \(x_0 \in \Omega\) and \(M > 0\). At this point, to reach a contradiction we will show that \(z(x_0) > M\). Suppose \(z(x_0) \leq M\). For all \(\delta > 0\), we set

\[
\Omega_\delta = \{x \in \Omega \mid \text{dist}(x, \partial \Omega) > \delta\}.
\]

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Since \( z(x) \to \infty \) as \( \text{dist}(x, \partial \Omega) \to 0 \), we can choose \( \delta > 0 \) such that \( z(x) > M \) for all \( x \in \Omega \setminus \Omega_{\delta} \). Obviously, \( x_0 \in \Omega_{\delta} \). Moreover, \( M - z(x_0) \geq 0 \) and \( (M - z)|_{\partial \Omega_{\delta}} \leq 0 \). Therefore we can find \( \bar{x} \in \Omega_{\delta} \) such that
\[
\max_{\Omega_{\delta}} (M - z(x)) = M - z(\bar{x}) \leq 0.
\]
It follows that \( \Delta(M - z)(\bar{x}) \leq 0 \), that is \( \Delta z(\bar{x}) \geq 0 \) which contradicts (7). Hence (2) has no positive large solutions. This completes the proof.  

**Remark 3** We can employ the same method as above to show that the system
\[
\begin{align*}
\Delta u + |\nabla v| &= p(|x|) f(v) \quad \text{in } \Omega, \\
\Delta v + |\nabla u| &= q(|x|) g(u) \quad \text{in } \Omega,
\end{align*}
\]
has no positive large solutions if \( f \) and \( g \) satisfy \((A_1)\).  

### 3. Proof of Theorem 2

Arguing by contradiction, let us assume that the system (2) has the positive entire large solution \((u, v)\).
Consider the spherical average of \( u \) and \( v \) defined by
\[
\begin{align*}
\bar{u}(r) &= \frac{1}{c_N r^{N-1}} \int_{|x|=r} u(x) d\sigma_x, \quad r \geq 0 \quad (9) \\
\bar{v}(r) &= \frac{1}{c_N r^{N-1}} \int_{|x|=r} v(x) d\sigma_x, \quad r \geq 0 \quad (10)
\end{align*}
\]
where \( c_N \) is the surface area of the unit sphere in \( \mathbb{R}^N \). Since \( u \) and \( v \) are positive entire large solutions it follows that \( \bar{u}, \bar{v} \) are positive and
\[
\lim_{r \to \infty} \bar{u}(r) = \lim_{r \to \infty} \bar{v}(r) = +\infty.
\]
By the change of variable \( x \to ry \), we have
\[
\bar{u}(r) = \frac{1}{c_N} \int_{|y|=1} u(ry) \, d\sigma_y, \quad r \geq 0
\]
and
\[
\bar{u}'(r) = \frac{1}{c_N} \int_{|y|=1} \nabla u(ry) \cdot y \, d\sigma_y, \quad r \geq 0. \quad (11)
\]
The above relation may be rewritten as
\[
\bar{u}'(r) = \frac{1}{c_N} \int_{|y|=1} \frac{\partial u}{\partial \rho}(ry) \, d\sigma_y = \frac{1}{c_N r^{N-1}} \int_{|x|=r} \frac{\partial u}{\partial \rho}(x) \, d\sigma_x,
\]
that is
\[
\bar{u}'(r) = \frac{1}{c_N r^{N-1}} \int_{|x|=r} \Delta u(x) \, d\sigma_x, \quad \text{for all } r \geq 0. \quad (12)
\]
Similarly we have
\[
\bar{v}'(r) = \frac{1}{c_N r^{N-1}} \int_{|x|=r} \Delta v(x) \, d\sigma_x, \quad \text{for all } r \geq 0. \quad (13)
\]
Due to the presence of the gradient term in (2), we cannot infer that $\Delta u \geq 0$ in $\mathbb{R}^N$ and so we do not know if $\bar{u}' \geq 0$ (or $\bar{v}' \geq 0$) in $[0, \infty)$. In order to overcome this lack of monotonicity, set

$$U(r) = \max_{\bar{u} \leq r} \bar{u}, \quad V(r) = \max_{\bar{\nu} \leq r} \bar{\nu}.$$  \hspace{1cm} (14)

Now it is easy to see that $U, V$ are positive and non-decreasing functions. Moreover $U \geq \bar{u}, \ V \geq \bar{\nu}$ and $U(r), V(r) \to +\infty$ as $r \to \infty$.

By (A1), that there exists $M > 0$ such that

$$\max \left\{ f(t), g(t) \right\} \leq M(1 + t), \quad \text{for all} \ t \geq 0.$$  \hspace{1cm} (15)

Now (11), (12) and (15) lead to

$$\bar{u}'' + \frac{N - 1}{r} \bar{u}' + \bar{u}' \leq \frac{1}{c_N r^{N-1}} \int |\Delta u(x) + |\nabla u(x)| \, d\sigma_x$$

$$= \frac{p(r)}{c_N r^{N-1}} \int f(v(x)) \, d\sigma_x$$

$$\leq \frac{M p(r)}{c_N r^{N-1}} \int (1 + v(x)) \, d\sigma_x$$

$$= M p(r) (1 + \bar{\nu}(r))$$

$$\leq M p(r) (1 + V(r)),$$

for all $r \geq 0$. It follows that

$$(r^{N-1} e^r \bar{u}')' \leq M e^r r^{N-1} p(r) (1 + V(r)) \quad \text{for all} \ r \geq 0.$$  \hspace{1cm} (16)

So, for all $r \geq r_0 > 0$,

$$\bar{u}(r) \leq \bar{u}(r_0) + M \int_{r_0}^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} p(s) (1 + V(s)) \, ds \, dt$$

$$\leq \bar{u}(r_0) + M \int_{r_0}^r e^{-t} t^{1-N} (1 + V(t)) \int_0^t e^s s^{N-1} p(s) \, ds \, dt$$

$$\leq \bar{u}(r_0) + M (1 + V(r)) \int_{r_0}^r e^{-t} t^{1-N} \int_0^t e^s s^{N-1} p(s) \, ds \, dt,$$

that is

$$\bar{u}(r) \leq \bar{u}(r_0) + M (1 + V(r)) \int_{r_0}^r P(t) \, dt, \quad \text{for all} \ r \geq r_0 \geq 0.$$  \hspace{1cm} (16)

Since $\int_1^\infty P(r) \, dr < \infty$ and $\int_1^\infty Q(r) \, dr < \infty$, we can choose $r_0 \geq 1$ such that

$$\max \left\{ \int_{r_0}^\infty P(r) \, dr, \int_{r_0}^\infty Q(r) \, dr \right\} < \frac{1}{2M}.$$  \hspace{1cm} (17)

From (14) and the fact that $\lim_{r \to \infty} \bar{u}(r) = \lim_{r \to \infty} \bar{\nu}(r) = \infty$, we can find $r_1 \geq r_0$ such that

$$U(r) = \max_{r_0 \leq r \leq r_1} \bar{u}(r), \quad V(r) = \max_{r_0 \leq r \leq r_1} \bar{\nu}(r), \quad \text{for all} \ r \geq r_1.$$  \hspace{1cm} (18)

Thus (16) and (18) yield

$$U(r) \leq \bar{u}(r_0) + M (1 + V(r)) \int_{r_0}^r P(t) \, dt, \quad \text{for all} \ r \geq r_1.$$  \hspace{1cm} (16)
Furthermore, by (17) we obtain
\[ U(r) \leq \bar{u}(r_0) + \frac{1 + V(r)}{2} \text{ for all } r \geq r_1, \]
and so
\[ U(r) \leq C_1 + \frac{1}{2}V(r) \text{ for all } r \geq r_1, \quad (19) \]
where \( C_1 = \frac{1}{2} + \bar{u}(r_0) > 0 \). In a similar way we get
\[ V(r) \leq C_2 + \frac{1}{2}U(r) \text{ for all } r \geq r_1, \quad (20) \]
By addition, (19) and (20) lead to
\[ U(r) + V(r) \leq 2(C_1 + C_2) \text{ for all } r \geq r_1. \quad (21) \]
This means that \( U \) and \( V \) are bounded and so \( u \) and \( v \) are bounded which is a contradiction. It follows that (2) has no positive entire large solutions and the proof is now complete. □

4. Proof of Theorem 3

We start by showing that (2) has positive radial solutions. On this purpose we fix \( a > 0 \) and \( b > 0 \) and we show that the system
\[
\begin{cases}
  u'' + \frac{N-1}{r}u' + u' = p(r)f(v(r)), & r > 0, \\
v'' + \frac{N-1}{r}v' + v' = q(r)g(u(r)), & r > 0, \\
u', v' \geq 0 & \text{on } [0, \infty), \\
u(0) = a > 0, & v(0) = b > 0,
\end{cases} \quad (22)
\]
has solutions. Then \( U(x) = u(|x|), V(x) = v(|x|) \) are positive solutions of (2).

Integrating (22) we have
\[ u(r) = a + \int_0^r e^{-t^{1-N}} \int_0^t e^s s^{N-1} p(s) f(v(s)) ds dt \quad \forall r \geq 0, \quad (23) \]
\[ v(r) = b + \int_0^r e^{-t^{1-N}} \int_0^t e^s s^{N-1} q(s) g(u(s)) ds dt \quad \forall r \geq 0. \quad (24) \]
Define \( v_0 \equiv b \) and let \((u_k)_{k \geq 1}, (v_k)_{k \geq 1}\) given by
\[ u_k(r) = a + \int_0^r e^{-t^{1-N}} \int_0^t e^s s^{N-1} p(s) f(v_{k-1}(s)) ds dt \quad \forall r \geq 0, \quad (25) \]
\[ v_k(r) = b + \int_0^r e^{-t^{1-N}} \int_0^t e^s s^{N-1} q(s) g(u_k(s)) ds dt \quad \forall r \geq 0. \quad (26) \]
Since \( v_1(r) \geq b \), it follows that \( u_2(r) \geq u_1(r) \) for all \( r \geq 0 \) which yields \( v_2(r) \geq v_1(r) \) and so \( u_5(r) \geq u_2(r) \) for all \( r \geq 0 \). Repeating such arguments we deduce that
\[ u_k(r) \leq u_{k+1}(r) \text{ and } v_k(r) \leq v_{k+1}(r), \quad \text{for all } r > 0, k \geq 1. \]
Let us now prove that the non-decreasing sequences \((u_k)_{k \geq 1}\) and \((v_k)_{k \geq 1}\) are bounded from above on bounded sets. We first observe that (25) and (26) yield
\[
u_k(r) \leq u_{k+1}(r) \leq a + f(v_k(r)) \int_0^r P(t) dt, \quad \forall r \geq 0, \ k \geq 1
\]
and
\[
u_k(r) \leq b + g(u_k(r)) \int_0^r Q(t) dt, \quad \forall r \geq 0, \ k \geq 1
\]
Let \(R > 0\) be arbitrary. From (27) and (28) we get
\[
u_k(R) \leq a + f \left( b + g(u_k(R)) \int_0^R Q(t) dt \right) \int_0^R P(t) dt, \quad \forall k \geq 1.
\]
This imply
\[
1 \leq \frac{a}{u_k(R)} + \frac{f \left( b + g(u_k(R)) \int_0^R Q(t) dt \right)}{u_k(R)} \int_0^R P(t) dt, \quad \forall k \geq 1.
\]
Replacing \(r = R\) in (27) and the assumption \((A_2)\) leads us to a contradiction. Thus \(L(R)\) is finite. Since \(u_k, v_k\) are increasing functions, it follows that the map \((0, \infty) \ni R \mapsto L(R)\) is non-decreasing on \((0, \infty)\) and
\[
u_k(r) \leq u_k(r) \leq L(R), \quad \forall r \in [0, R], \forall k \leq 1,
\]
\[
u_k(r) \leq b + g(L(R)) \int_0^r Q(t) dt, \quad \forall r \in [0, R], \forall k \leq 1.
\]
Furthermore, there exists \(\lim_{R \to \infty} L(R) = \bar{L} \in (0, \infty]\) and the sequences \((u_k)_{k \geq 1}\) and \((v_k)_{k \geq 1}\) are bounded from above on bounded sets.

Let \(u(r) := \lim_{k \to \infty} u_k(r), \ v(r) := \lim_{k \to \infty} v_k(r)\) for all \(r \geq 0\). By standard elliptic regularity theory we deduce that \((u, v)\) is a positive solution of (22).

In order to conclude the proof, it is enough to show that \((u, v)\) is a large solution of (22). Let us remark that (23), (24) imply
\[
u(r) \geq a + f(b) \int_0^r P(t) dt, \quad \forall r \geq 0,
\]
\[
u(r) \geq b + g(a) \int_0^r Q(t) dt, \quad \forall r \geq 0.
\]
Since \(f, g\) are positive functions and \(p, q\) satisfy (5) we can conclude that \((u, v)\) is a large solution of (22) and \((U, V)\) is a positive entire large solution of (2). Hence any large solution of (22) provides a positive entire large solution \((U, V)\) of (2) with \(U(0) = a\) and \(V(0) = b\). Since \((a, b) \in (0, \infty) \times (0, \infty)\) was chosen arbitrarily, it follows that (2) has infinitely many positive entire large solutions. The proof of theorem is now complete. \(\blacksquare\)

Remark 4 The condition (5) is sufficient but not necessary for the existence of positive entire large solutions for (2). Indeed, let us consider \(f(t) = \sqrt{t}, \ g(t) = t, \ p(r) = 4 \frac{r^3 + (N + 2)r^2}{\sqrt{r^2 + 1}}, \ q(r) = 2 \frac{r + N}{r^4 + 1} \).
Using (6) we get
\[
\int_1^{\infty} P(r) dr = +\infty \quad \text{and} \quad \int_1^{\infty} Q(r) dr < +\infty.
\]
However, the corresponding system to (2) is
\[
\begin{aligned}
\Delta u + |\nabla u| &= 4 \frac{|x|^3 + (N + 2)|x|^2}{\sqrt{|x|^2 + 1}} \cdot \sqrt{7} \quad \text{in } \mathbb{R}^N, \\
\Delta v + |\nabla v| &= 2 \frac{|x| + N}{|x|^4 + 1} \cdot u \quad \text{in } \mathbb{R}^N,
\end{aligned}
\]
which has the positive entire large solution \((|x|^4 + 1, |x|^2 + 1)\). ■

References


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