A note about a priori estimates for indefinite problems in unbounded domains

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Abstract. In this paper, we are dealing with the following superlinear elliptic problem:

\[(P_{\Omega}) \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega, \ u \geq 0 \end{cases} \]

where \(\Omega\) a smooth domain not necessarily bounded, \(h\) is a \(C^2\) function from \(\mathbb{R}^N\) to \(\mathbb{R}\) changing sign such that \(h(x) \to -\infty\) when \(\|x\| \to +\infty\) and \(1 < p < \frac{N+2}{N-2}\). We give existence and uniform a priori estimates for solutions to \((P_{\Omega})\).

Una nota sobre estimaciones a priori para problemas indefinidos en dominios no acotados

Resumen. Consideramos el siguiente problema elíptico superlineal

\[(P_{\Omega}) \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{en } \Omega \\ u = 0 & \text{en } \partial\Omega, \ u \geq 0 \end{cases} \]

donde \(\Omega\) es un dominio regular no necesariamente acotado, \(h\) es una función \(C^2\) de \(\mathbb{R}^N\) en \(\mathbb{R}\) cambiando de signo tal que \(h(x) \to -\infty\) cuando \(\|x\| \to +\infty\) y \(1 < p < \frac{N+2}{N-2}\). Obtenemos la existencia y estimaciones a priori de las soluciones de \((P_{\Omega})\).

1. Introduction

In this paper, we consider the following superlinear elliptic problem:

\[(P_{\Omega}) \begin{cases} -\Delta u = \lambda u + h(x)u^p & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega = 0 \end{cases} \]

Our goal is to extend the results in [8] where \((P_{\Omega})\) is also investigated in the case \(\Omega = \mathbb{R}^N\). Precisely, in [8] assuming that \(\Omega^+ = \{x \in \mathbb{R}^N / h(x) > 0\}\) is a bounded domain, that \(\Gamma := \{x \in \mathbb{R}^N / h(x) = 0\}\) satisfies a nondegeneracy condition:

\[\forall x \in \Gamma, \nabla h(x) \neq 0,\]
the authors prove that there exist a global branch bifurcating from the essential spectrum, in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. For this, they prove that for $\lambda$ bounded, the solutions to $(P_{\lambda, k})$ obtained by a local approach are uniformly bounded in $L^\infty(\mathbb{R}^N)$. The method they use involves studying a “local problem”, $(P_{\lambda, k})$, in a bounded domain $\Omega_R \supset B_R$ where $B_R$ is the ball centered at 0 and with radius $R$

\begin{equation}
(P_{\lambda, k}) \begin{cases}
-\Delta u = \lambda u + h(x)u^p & \text{in } \Omega_R, \\
u \in H^1_0(\Omega_R), & u \geq 0,
\end{cases}
\end{equation}

and then they pass to the limit when $R$ goes to $+\infty$.

The crucial step in this procedure is to get a priori estimates for solutions to $(P_{\lambda, k})$ independent of $R$.

Here we prove that on some conditions, we can remove the nondegeneracy assumption : $h$ vanishes in a non zero measure set and get the same results as in [8] for any large domain $\Omega$ with smooth boundary. Furthermore, the a priori estimates we obtain concern all solutions to $(P_{\lambda, k})$. Note that a large class of unbounded domains are considered.

Here, we suppose that $h$ satisfies the following assumptions :

(H1) $h \in C^2([0, \infty) \times \mathbb{R})$, $\Omega^+ := \{x \in \mathbb{R}^N, h(x) > 0\}$ is bounded domain with non zero measure and smooth boundary.

Supposing that there exists $\Omega_0 := \{x \in \mathbb{R}^N / h(x) = 0\}/\partial \Omega^- \cup \partial \Omega^+$, we assume in addition

(H2) $\Omega_0$ is bounded with smooth boundary and $\partial \Omega_0 \cap \partial \Omega^- \cap \partial \Omega^+ = \emptyset$,

(H3) For $x$ close to $\partial \Omega_0 \cap \partial \Omega^+$, $h(x) \equiv C \operatorname{dist}(x, \Omega_0 \cap \partial \Omega^+)^\gamma, \gamma > 0, C > 0$.

Let $\Gamma := \partial \Omega^+ \cap \partial \Omega^-$, then if $\Gamma$ is non empty, $\Gamma$ satisfies either

(H4) for $x \in \Omega^+$ close to $\Gamma$, $h(x) \equiv C \operatorname{dist}(x, \Gamma)^\gamma', \gamma' > 0, C > 0$

or

(H4bis) for any $x \in \Gamma$, $\nabla h(x) \neq 0$.

Remark 1

1. Clearly (H1) and (H2) imply that $\Gamma = \overline{\Omega^+} \cap \overline{\Omega^-}$ and it is bounded.

2. (H2) implies that $\Omega_0$ is far from $\Gamma$.

3. (H3) and (H4) give some flatness condition on $h$ near $\partial \Omega_0$ and $\Gamma$. We use (H3) and (H4) in a blow up technique as in [6]. A similar argument is also used in [3].

4. (H4bis) is the nondegeneracy condition as in [8].

Our purpose is to prove the existence of solutions and to give the structure of solutions set with respect to the bifurcation parameter $\lambda$.

When $h(x)$ changes sign, the proof of existence of a priori estimates is more difficult to obtain. Let us mention some previous works in this direction :

In [6], the authors use a blow up technique combined with some Liouville theorems in cones to obtain a priori bounds and some existence results for equation $(P_{\lambda, k})$ with $\Omega$ a bounded domain for $1 < p < \frac{N+2}{N-2}$ and $\lambda = 0$. (H4bis) is used to get $\gamma' = 1$ in this paper. The question which follows then is : is it true for any $p$ less the critical exponent?

In [12] Chen and Li answer positively to that question i.e. they obtain some a priori bounds for positive solutions when $p$ is subcritical (i.e. $p < \frac{N+2}{N-2}$). Precisely they consider the following problem

\begin{equation}
\begin{cases}
-\Delta u = h(x)u^p & \text{in } \Omega, \\
u \in H^1_0(\Omega) & u \geq 0,
\end{cases}
\end{equation}

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where \( h \) satisfies (H1), (H4bis), \( \Omega_0 = \emptyset \) and \( \Gamma \subset \Omega \). They prove that every solution is uniformly bounded and that the a priori bound depends only on the geometry of \( \Omega, p \) and \( h \).

The proof of this result is carried out dividing the domain in three regions and then solving the following steps:

1. boundedness of solutions in the region where \( h(x) \leq -\delta \), for a fixed \( \delta > 0 \),
2. boundedness of solutions in the region where \(|h(x)|\) is small,
3. boundedness of solutions in the region where \( h(x) \geq \delta \).

Each step involves different techniques:

1. In the region where \( h(x) \) is strictly negative, the uniform estimate is obtained by an Harnack inequality and an integral estimate.
2. In the region where \(|h(x)|\) is small, the a priori bound results from the moving plane technique (here (H4bis) plays a crucial role) and from the above estimate.
3. In the last region, a classical blow up analysis (see [16]) is used.

In [3], the authors remove the nondegeneracy condition using the same blow-up technique as in [6] but they keep some restrictions on \( p \) due to the restriction in Liouville’s theorem they apply. In [19], the authors prove that if we restrict to some type of solutions (with finite Morse index solutions precisely) the restriction of \( p \) can be removed also in the degenerate case.

In the present work, combining different techniques, some of them booked from [6], [12] and [3], we get uniform a priori estimates in in bounded domains case or (if \( \Omega \) is bounded) independent of the measure of the domain considered and independent of \( \lambda \) bounded. Precisely, we prove the following main results:

**Theorem 1.** Suppose that (H1), (H2), (H3) are satisfied, that 
\[ 1 < p < \frac{N+2}{N-2} \] 
and that \( \Omega \) is large enough that \( \Gamma \cup \partial \Omega_0 \subset \Omega \) and \( \partial \Omega \subset \text{supp } h^- \). Let \( \lambda_1(\Omega^+) \) (resp. \( \lambda_1(\Omega_0) \)) be the first eigenvalue to \(-\Delta\) in \( \Omega^+ \) (resp. in \( \Omega_0 \)). We also assume that \( \Omega_0 \) is nonzero measure set. Then,

(i) If \( \lambda \geq \inf(\lambda_1(\Omega^+), \lambda_1(\Omega_0)) \), there are no non trivial solutions of \((P_{\Omega})\).

(ii) Assume in addition that \( \Gamma \) is nonempty and (H4). Let \( p \) such that \( 1 < p < \frac{N+1+\inf(\gamma, \gamma')}{N-1} \). For any \( \lambda_0 < \lambda_2 < \inf(\lambda_1(\Omega^+), \lambda_1(\Omega_0)) \), there is a constant \( C = C(\lambda_0, \lambda_2) \) such that if \( (\lambda, u) \) is a solution of \((P_{\Omega})\) and \( \lambda_0 \leq \lambda \leq \lambda_2 \) then

\[ \|u\|_{L^\infty} \leq C \]  

and \( C \) depends only on \( \lambda_0, \lambda_2, \Omega^+, \Omega_0, p \) and \( h \).

(iii) If \( \partial \Omega_0 \cap \partial \Omega^+ = \emptyset \) and (H4bis) holds, then (1) is also true for any \( p \) subcritical.

(iv) If \( \Gamma \) is empty or if (H4bis) holds instead of (H4), then (1) is true for \( 1 < p < \frac{N+1+\gamma'}{N-1} \) and \( C \) depends also on \( \Gamma \).

**Remark 2**

1) Theorem 1 concern the case where \( \Omega_0 \) is non zero measure set. If \( \Omega_0 \) is empty and if (H4bis) holds, then we can apply results in [8].

2) Theorem 1 handle unbounded domains \( \Omega \) as bounded domains. For unbounded domains, the dirichlet conditions are replaced by a limit condition at infinity.
Next, we show that if $h$ has radial symmetry properties and $\partial \Omega^+$ has only one piece component then no restriction on $p$ subcritical and no nondegeneracy condition are necessary. A similar observation was previously made in [3] for bounded domains.

**Proposition 1.** Assume that $h$ is radial symmetric continuous function and that $\Omega^+$ is a ball. Then, (1) is also true for radial symmetric solutions.

**Remark 3** If $\partial \Omega^+$ has two pieces component and if (H4bis) holds then Proposition 1 is also true.

Finally, in the next result we show that the asymptotic behaviour of $h$ is relevant to determine the behaviour of solutions to $(P)$.

**Proposition 2.** Assume that $h$ is continuous function on $\mathbb{R}^N$ such that
\[
\lim_{|x|\to+\infty} h(x) = a < 0 \text{ and finite.}
\]
Let $\lambda > 0$. Then, for any nontrivial solution $u$ to (P), we have
\[
\liminf_{|x|\to+\infty} u(x) > 0.
\]

**Remark 4**

1. Proposition 2 is also valid in the case where $h$ satisfies $h(x) < 0$ for $|x|$ large and $\lim_{|x|\to\infty} h(x) = 0$.

2. We can extend easily Proposition 2 in the case where $u^p$ is replaced by $g(u)$, $g \in C^2(\mathbb{R}^+)$ satisfying $s \to g(s)$ nonincreasing, $\lim_{s \to 0^+} g(s) = 0$ and $\lim_{s \to +\infty} \frac{g(s)}{s} = +\infty$.

Using the above a priori estimates and the global bifurcation Rabinowitz (see [18]) Theorem, we get existence of solutions to $(P)$ and the behaviour of solutions with respect to the bifurcation parameter $\lambda$. Consider $\phi_{\Omega} > 0$ the eigenfunction associated to the first eigenvalue $\lambda_1(\Omega)$ which satisfies:
\[
\begin{cases}
-\Delta \phi_{\Omega} = \lambda_1(\Omega) \phi_{\Omega} & \text{in } \Omega \\
\phi \geq 0.
\end{cases}
\]
and $\int_{\Omega} \phi_{\Omega} = 1$. Let $\Pi_{\mathbb{R}}$ denote the projection onto $\mathbb{R}$. We will prove as application of a priori estimates the following results:

**Theorem 2.**
Assume that the assumptions of Theorem 1 are satisfied. We have the following:
If $\lambda_1(\Omega^+) < \lambda_1(\Omega_0)$, then there is a global branch of nontrivial solutions of $(P)$, $C$, connected in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$, bifurcating from $(0, 0)$ such that
\[(i) \quad \Pi_{\mathbb{R}} C = ]-\infty, \lambda_0[ \text{ where } 0 < \lambda_0 < \lambda_1(\Omega^+).\]

\[(ii) \quad \text{Let } (\lambda_n, u_n) \in C \text{ such that } \lambda_n \to -\infty \text{ as } n \to +\infty. \text{ Then, up to subsequences, } \|u_n\|_{H^1, L^\infty} \to +\infty.\]

Finally, if $\Omega^+$ is empty, then we have:

**Theorem 3.**
Assume that $\Omega^+ = \emptyset$ and $\Omega_0$ bounded.

\[(i) \quad \text{if } \Omega_0 \text{ is a nonzero measure set, then there exists a global branch of nontrivial solutions of } (P), C, \text{ connected in } \mathbb{R} \times L^\infty(\mathbb{R}^N), \text{ bifurcating from } (0, 0) \text{ such that} \]
\[a) \quad \Pi_{\mathbb{R}} C = ]0, \lambda_1(\Omega_0)[,\]

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Let \((\lambda_n, u_n) \in C\) such that \(\lambda_n \to \lambda_1(\Omega_0)\) as \(n \to +\infty\). Then, up to subsequences, \(||u_n||_{L^\infty} \to +\infty||\)

(ii) If \(\Omega_0\) is empty, then (i) is also valid replacing \(\lambda_1(\Omega_0)\) by \(+\infty\).

The outline of the paper is as follows:

In Section 2, We prove the results concerning a priori estimates. Theorem 1, Proposition 1, Proposition 2.
In Section 3, we prove Theorems 2 and 3.


First, we prove Theorem 1:

**Proof of Theorem 1.**

Let us prove (i). We use a standard argument for superlinear elliptic problems. Multiply \((P_\Omega)\) by \(\phi_{\Omega^+}\) and integrate by parts in \(\Omega^+\), we obtain:

\[
\lambda_1(\Omega^+) \int_{\Omega^+} u \phi_{\Omega^+} + \int_{\Omega^+} \frac{\partial \phi_{\Omega^+}}{\partial n} u = \int_{\Omega^+} h(x) u^p \phi_{\Omega^+} + \lambda \int_{\Omega^+} u \phi_{\Omega^+}. \tag{3}
\]

From (3), and Hopf lemma:

\[
(\lambda_1(\Omega^+) - \lambda) \int_{\Omega^+} u \phi_{\Omega^+} \geq \int_{\Omega^+} h(x) u^p \phi_{\Omega^+} > 0,
\]

which implies that \(\lambda < \lambda_1(\Omega^+)\). Repeating the argument with \(\phi_{\Omega_0}\), (i) is proved. Let us prove (ii). For this, we divide the proof in several parts:

1. Local Estimates in \(\Omega^+_\delta := \{ x \in \overline{\Omega} \cap \text{supph}^- / d(x, \Gamma \cup \Omega_0) \geq \delta \}\). The estimate is obtained as in [8] by a uniform local \(L^p\)-estimate + the Harnack inequality (see proof of Proposition 1.1 step 1 in [8]).

2. Estimates in \(\Omega^+\). We use a blow up technique as in [6] and [3].

3. Estimates in \(\Omega_0\). We use a super and sub solutions argument.

4. \(L^\infty\)-bound for \(|x|\) large, obtained by the construction of a supersolution.

**Step 1:** A priori estimates in \(\Omega^+_\delta\). See [8] or [14] (Proposition 5.1 and step 1 in the proof of Proposition 5.2). Note that since \(\partial \Omega \subset \text{supph}^-\) and by maximum principle we get a priori bounds near \(\partial \Omega\).

**Step 2:** A priori estimates in \(\Omega^+\). For this, suppose by contradiction that there exists a sequence \((\lambda_k, u_k)\) solution to \((P)\) with \(\lambda_0 \leq \lambda_k \leq \lambda_2\) and \(\sup u_k \to +\infty\). Let \(x_k\) such that \(u_k(x_k) = \sup_{\Omega^+} u_k\). Up to a subsequence, we can suppose that \(x_k \to x_0 \in \overline{\Omega^+}\).

We now deal with two cases:

First, suppose that \(x_0 \in \Omega^+\) then we can conclude using [16] and get the contradiction. Suppose now that \(x_0 \in \partial \Omega^+\). Then, either \(x_0 \in \partial \Omega^+ \cap \partial \Omega_0\) or \(x_0 \in \partial \Omega^+ \cap \partial \Omega^-\). Now following the same blow up analysis in [3] (see theorem 4.3) and using (H3) in the case \(x_0 \in \partial \Omega^+ \cap \partial \Omega_0\) (resp. (H4) in the case \(x_0 \in \Gamma\)) we get a contradiction by a Liouville theorem in cones (see Theorem 2.1 in [6]).

**Step 3:** A priori estimates in \(\Omega_0\). Let \(\Omega^+_0\) one of the connected component of \(\Omega_0\). By (H2), \(\partial \Omega^+_0\) belongs to \(\partial \Omega^-\) or \(\partial \Omega^+\). Suppose that \(\partial \Omega^+_0\) belongs to \(\partial \Omega^-\), then we construct a supersolution in a \(\varepsilon\)-neighborhood of \(\Omega_0\), denoted by \(\Omega_\varepsilon\) : Let \(\lambda_\varepsilon\) such that \(\lambda_2 < \lambda_\varepsilon < \lambda_1(\Omega_0)\) and \(\xi_\varepsilon\) the solution to

\[
\begin{cases} 
-\Delta \phi = \lambda_\varepsilon \phi & \text{in } \Omega_\varepsilon \\
\phi = M & \text{in } \partial \Omega_\varepsilon.
\end{cases}
\]
The existence and uniqueness of $\xi$ is provided by the fact that $\lambda_r < \lambda_1(\Omega_0)$ and $\epsilon$ small enough such that $\lambda_r < \lambda_1(\Omega_r)$.

Next, we choose $M = \sup u$ which do not depend on $u$ since in Step 1 we have proved uniform local a priori estimates in $\Omega^-_r$ (note that $\partial \Omega_r$ belongs to $\Omega^-_r$ for an appropriated $\delta$). Then, by maximum principle, we have $u \leq \xi_r$.

Now, consider the case $\partial \Omega_0^1 \subset \partial \Omega^+$. Therefore, by step 2, we have that $u$ is uniformly bounded on $\partial \Omega_0^1$. Let $M$ be the uniform bound of $u$ in $\partial \Omega_0^1$. Now, consider $\xi$ the unique solution to

$$
\begin{cases}
-\Delta \xi = \lambda_2 \xi & \text{in } \Omega_0^1 \\
\phi = M & \text{in } \partial \Omega_0^1.
\end{cases}
$$

Therefore, by the maximum principle, $u \leq \xi$ in $\Omega_0^1$. Finally, we get an uniform bound in $\Omega_0$ since $\Omega_0$ is bounded and have only finitely many components.

**Step 4:** A priori estimates for $|x|$ large. This part concerns the case where $\Omega$ is unbounded and we can suppose here that $\Omega = \mathbb{R}^N$. Let $R_0$ be such that $\{ \Gamma_\cup \partial \Omega_0 \} \subset B_{R_0}$ and $\phi$:

$$
(P_*) \begin{cases}
-\Delta \phi = \lambda_2 \phi + h^*(x) \phi^p & \text{in } \mathbb{R}^N / B_{R_0} \\
\phi = M & \text{in } \partial B_{R_0}, \phi \to 0 \text{ when } |x| \to +\infty.
\end{cases}
$$

From a priori estimates in step 1, we choose $M$ such that for any solution $u$, $\sup_{\partial B_{R_0}} u(x) \leq M$ and $h^*$ a continuous function such that $h^* \geq h$ we fix later. Then, by the maximum principle, $u \leq \phi$. Next, thanks to $\lim_{|x| \to \infty} h(x) = -\infty$ we prove that $\phi(x)$ tends to $0$ when $|x| \to +\infty$. For this, we choose $h^*$ negative, radial symmetric, decreasing for large $r = |x|$ and $h^*(r) \to -\infty$ when $r \to +\infty$.

To prove the existence of $\phi$, we consider the following problem :

$$
(P_{R_*}) \begin{cases}
-\Delta \phi = \lambda_2 \phi + h^*(x) \phi^p & \text{in } B_{R_0} / B_{R_0} \\
\phi = M & \text{in } \partial B_{R_0} \text{ and } \phi = 0 \text{ in } \partial B_R.
\end{cases}
$$

For $R$ large, we claim that there exists a unique solution to $(P_{R_*})$. For this, consider $\psi_M$ a smooth continuation of $M$ with compact support and the following minimization problem:

$$
I_R = \min_{v \in \mathcal{H}_0^1(B_R < |x| < R)} \mathcal{E}(v) := \frac{1}{2} \int_{R_0 < |x| < R} (|\nabla (v + \psi_M)|^2 - \lambda_2 (v + \psi_M)^2) \\
+ \frac{1}{p + 1} \int_{R_0 < |x| < R} |h^*(v + \psi_M)^{p+1}|
$$

By Sobolev imbeddings, we get easily $I_R > -\infty$ then, a global minimizer solution $\phi_R$ to $(P_{R_*})$ exists. The uniqueness is a standard argument using the concavity of the nonlinearity (see appendix II in [9]). By uniqueness of $\phi_R$ and doing $R \to +\infty$, we get a minimal solution $\phi$ to $(P_*)$ and $\phi$ is radial. Note that $\phi$ is bounded and if $x_0$ is a local maximum, then $\phi(x_0) \leq \left( \frac{\lambda_2}{h^*(|x_0|)} \right)^{p-1}$. Therefore, since $h^*(r) \to -\infty$ when $r \to +\infty$, either $\phi$ is decreasing for $|x|$ large either $\phi(x) \to 0$ when $|x| \to +\infty$. Assume that the first possibility holds then $\phi(x) \to 0$ when $|x| \to +\infty$. If $I \neq 0$ then the O.D.E satisfied by $\phi$ shows that $\phi''(r) \to +\infty$ when $r \to +\infty$ which is impossible since $\phi$ is bounded. Note that we need here the local approach. Indeed, using that $\psi_M$ has a compact support, for $R$ large we claim that $I_R \to -\infty$ when $R \to +\infty$. For this, define $v_R$ as follows :

$$
v_R(x) = \begin{cases}
0 & \text{if } |x| \leq \frac{R}{2} - 1 \text{ or } |x| \geq R + 1 \\
\left( \frac{\lambda_2}{h^*} \right)^{p-1} & \text{if } \frac{R}{2} \leq |x| \leq R \\
\left( \frac{\lambda_2}{h^*} \right)^{p-1} (|x| - \frac{R}{2} + 1) & \text{if } \frac{R}{2} - 1 \leq |x| \leq \frac{R}{2} \\
\left( \frac{\lambda_2}{h^*} \right)^{p-1} (|x| - R) & \text{if } R \leq |x| \leq R + 1
\end{cases}
$$
Now, observing that by a simple computation
\[
\mathcal{E}(v_R - \psi_M) \leq C_1 \left( \frac{\lambda^2}{R^\alpha} \right)^{\frac{2}{p+1}} R^{N-1} - C_2 \left( \frac{\lambda^2}{R^\alpha} \right)^{\frac{2}{p+1}} R^N
\]
which implies that \( \mathcal{E}(v_R - \psi_M) \rightarrow -\infty \) when \( R \rightarrow +\infty \), we have for \( R \) large enough
\[
\mathcal{E}(v_R) = \mathcal{E}(v_R - \psi_M) + \mathcal{E}(0) \rightarrow -\infty \) when \( R \rightarrow +\infty \).

This completes the proof of assertion (ii).

(iii) and (iv) are a direct application of step 2 in [8] to get an a priori bound in a neighborhood of \( \Gamma \) instead of the blow up analysis when \( x_0 \in \Gamma \). The proof of Theorem 1 is now complete. □

Now, let us deal with the case where \( h \) is radial symmetric.

**Proof of Proposition 1:**

Note first that if \( \Omega^+ \) is a ball \( B_{R_1} \), then by the maximum principle, for any solution \( u \) to (P1), we have for \( r \leq R_1 \), \( \min_{B_r} u(x) \) is attained on \( \partial B_r \). Furthermore, using the \( L^p \)-estimate in [8] (see p 23-24 or Proposition 5.1 in [14]), we have:
\[
\int_{B_r^\pm} u^p \leq C := C(r, \inf_{B_r} h, \phi_1(B_{R_1}), \lambda_1, \lambda_2, p).
\]
which implies that
\[
\min_{B_{R_1}} u(x) \leq \min_{B_r^\pm} u(x) \leq C := C(r, \inf_{B_r} h, \phi_1(B_{R_1}), \lambda_1, \lambda_2, p).
\]
Since \( u \) is radial symmetric, then \( u \) is uniformly bounded on \( \partial \Omega^+ \). Therefore doing again the blow up analysis as in step 2 of the proof of Theorem 1, we have that \( x_0 \in \Omega^+ \) and results in [16] are sufficient to get the contradiction. This completes the proof of Proposition 1. □

Finally, let us prove Proposition 2

**Proof of Proposition 2:**

Let \( (\lambda, u) \) a non trivial solution to (P) and \( R_0 \) such that \( \overline{\Omega^+ \cup \Omega^-} \subset B_{R_0} \). By Harnack inequality (see p. 199 in [17]), \( \min_{B_{R_0}} u(x) = m_u > 0 \). For \( R >> R_0 \), consider the following problem:
\[
(P_R) \left\{ \begin{array}{l}
-\Delta \phi_R = \lambda \phi_R - b \phi_R^p \quad \text{in } B_R \\
\phi = 0 \quad \text{on } \partial B_R,
\end{array} \right.
\]
where \( b > \sup_{|x| \geq R_0} |h(x)| \) and \( \left( \frac{\lambda}{b} \right)^{\frac{1}{p-1}} < m_u \). Then, by maximum principle, for all \( R \) large enough, it is easy to prove that \( \phi_R < u \) in \( B_R / B_{R_0} \) (since \( \sup \phi_R < \left( \frac{\lambda}{b} \right)^{\frac{1}{p-1}} \)). Now, we will show that \( \phi_R \rightarrow \left( \frac{\lambda}{b} \right)^{\frac{1}{p-1}} \) in \( L^\infty_{\text{loc}}(\mathbb{R}^N) \) when \( R \rightarrow +\infty \) which completes the proof of Proposition 2. For this, note that \( \phi_R \) is the unique nontrivial solution to \( (P_R) \) (see [5]) for \( R \) large. From [15] results, \( \phi_R \) is radial symmetric and decreasing. Then, \( \phi_R \) is also the global minimizer to
\[
I_R = \min_{v \in H_0^1(B_R)} \mathcal{E}(v) := \frac{1}{2} \int_{B_R} |\nabla v|^2 - \lambda v^2 + \frac{1}{p+1} \int_{B_R} b v^{p+1}
\]
and \( \mathcal{E}(\phi_R) \rightarrow -\infty \) when \( R \rightarrow +\infty \). For this, consider the following testing function:
\[
k_R(x) = \begin{cases}
0 & \text{if } |x| \geq R - 1 \\
\left( \frac{\lambda}{b} \right)^{\frac{1}{p-1}} & \text{if } R - 1 \leq |x| \\
\left( \frac{\lambda}{b} \right)^{\frac{1}{p-1}}(|x| - R + 1) & \text{if } R - 1 \leq |x| \leq R.
\end{cases}
\]
In this section, we prove Theorems 2 and 3.

If $R < R'$, then $\phi_R < \phi_{R'} < (\frac{\lambda}{b})^{\frac{1}{2-p}}$. Consequently, $\phi_R \rightarrow v$ when $R \rightarrow +\infty$ in $L^\infty_{loc}(\mathbb{R}^N)$. $v > 0$ is also radial symmetric and decreasing and satisfies:

$$-v_{rr} - (N-1)\frac{v_r}{r} = \lambda v - bv^p \text{ in } (0, +\infty)$$

Suppose that $v \neq (\frac{\lambda}{b})^{\frac{1}{2-p}}$, then using the above equation, it is easy to show that $v(+\infty) = 0$ and since $\phi_R \leq v$, $\phi_R$ tends uniformly to 0 when $|x| \rightarrow +\infty$. Then for all $\epsilon$, there exists $R_{\epsilon}$ such that $\phi_R(x) < \epsilon$ if $|x| \geq R_{\epsilon}$. Therefore, since $\phi_R$ is a solution to $(P_R)$

$$E(\phi_R) \leq -\left(1 - \frac{1}{p+1}\right)\int_{B_R} b|\phi_R|^{p+1} \geq -C(R_{\epsilon}) - \epsilon^{p-1}\int_{B_R} |\phi_R|^2 \geq -C(R_{\epsilon}) - \epsilon^{p-1}(\frac{\lambda}{b})^{\frac{2}{2-p}} R^N.$$

But, from (5) and from $\epsilon$ small enough, we get the contradiction for large $R$. This completes the proof of Proposition 2.  

**Remark 5** Note that the result is not valid if $\lambda \leq 0$. Indeed, for $K$ large, the function $K|x|^{\frac{2}{p-1}}$ is a supersolution to $(P_\ast)$ (see proof of Theorem 1 step 4). In [13], the authors investigate the decay of weak solutions of such problems.

### 3. Applications

In this section, we prove Theorems 2 and 3.

**Proof of Theorem 2**:

Since $\inf \lambda_1(\Omega_0), \lambda_1(\Omega^+) = \lambda_1(\Omega^+)$, we get from Theorem 1 that any nontrivial solution to $(P)$ (resp. $(P_\Omega)$) is uniformly bounded in $L^\infty(\mathbb{R}^N)$ (resp. in $L^\infty(\Omega)$). Then, we can proceed exactly as in [8] and get the same results. Note that the compactness of solutions in $L^\infty(\mathbb{R}^N)$ is provided by the uniform decay of solutions at infinity (see step 4 in the proof of Theorem 1).

Now, let us consider the case where $\Omega^+ = \emptyset$.

**Proof of Theorem 3**:

Let us prove (i). The existence of a global unbounded branch of solutions to $(P)$, $C$, is obtained exactly as in Theorem 2. So we don’t repeat the arguments (see [8] for details). To prove the global behaviour of $C$, note that if $(\lambda, u) \in C$, then $0 < \lambda < \lambda_1(\Omega_0)$. Indeed, suppose by contradiction that there exists $(\lambda, u) \in C$ with $0 > \lambda$, then since $u \rightarrow 0$ when $|x| \rightarrow +\infty$, by the maximum principle, $u \equiv 0$.

Therefore, $C$ has asymptotic bifurcation points. Using the a priori estimates in Theorem 1, we have that for $\lambda \neq \lambda_1(\Omega_0)$, the solutions $u$ are uniformly bounded. Consequently, there is one and only one bifurcation point $\lambda = \lambda_1(\Omega_0)$. This completes the proof of assertion (i).

Finally, to prove (ii), we should remark that if $\Omega_0 = \emptyset$, then $\lambda_1(\Omega_0) = +\infty$. Furthermore, when $\lambda \rightarrow +\infty$, $u(\lambda) \rightarrow +\infty$ in any compact $K \subset \mathbb{R}^N$. For this, note that if $\lambda_1 < \lambda_2$ then $u(\lambda_1) < u(\lambda_2)$. This completes the proof of Theorem 3.
Remark 6 In Assertion (ii) of Theorem 3, the branch $C$ is a smooth curve. Indeed, it is easy to prove that for $\lambda$ fixed, there is a unique non trivial solution to $(P)$ (the proof is the same as in bounded domain, see [5]). Then using results in [10], we can prove that $C$ is $C^1$. ■

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