The structure of nonseparable Banach spaces with uncountable unconditional bases

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Abstract. Let $X$ be a Banach space with an uncountable unconditional Schauder basis, and let $Y$ be an arbitrary nonseparable subspace of $X$. If $X$ contains no isomorphic copy of $\ell_1(J)$ with $J$ uncountable then (1) the density of $Y$ and the weak*-density of $Y^*$ are equal, and (2) the unit ball of $X^*$ is weak* sequentially compact. Moreover, (1) implies that $Y$ contains large subsets consisting of pairwise disjoint elements, and a similar property holds for uncountable unconditional basic sets in $X$.

La estructura de los espacios de Banach no separables que tienen bases incondicionales no numerables

Resumen. Sea $X$ un espacio de Banach con una base incondicional de Schauder no numerable, y sea $Y$ un subespacio arbitrario no separable de $X$. Si $X$ no contiene una copia isomorfa de $\ell_1(J)$ con $J$ no numerable entonces (1) la densidad de $Y$ y la débil*-densidad de $Y^*$ son iguales, y (2) la bola unidad de $X^*$ es débil* sucesionalmente compacta. Además, (1) implica que $Y$ contiene subconjuntos grandes formados por elementos disjuntos dos a dos, y una propiedad similar se verifica para las bases incondicionales no numerables de $X$.

1 Introduction

Throughout this paper $X$ will denote a Banach space with an unconditional basis $(x_\gamma)_{\gamma \in \Gamma}$, where $\Gamma$ is an uncountable set, and $Y$ will be its nonseparable closed linear subspace. The best known examples of such spaces $X$ are $\ell_p(\Gamma)$, $1 \leq p < \infty$, and $c_0(\Gamma)$ (another examples are addressed in [10, 15, 20, 24]). By $\ell_1(\Gamma)$ we denote the space $\ell_1(\Gamma)$ with $\text{card}(\Gamma) = \aleph_1$.

This paper deals with the structure of nonseparable subspaces of $X$ whose study is motivated by the result below, included implicitly in the proofs of two results by Rodriguez-Salinas ([15, Proposition 2]) and Granero ([5, Proposition 1]):

(RSG) Let $Y$ be a nonseparable subspace of $X$ with $\chi(Y) = \chi^*(Y^*)$. Then $Y$ contains a set of the cardinality of $\chi(Y)$ consisting of elements of norm one with pairwise disjoint supports.

(Here and in what follows $\chi(Y)$ and $\chi^*(Y^*)$, respectively, denote the density and weak*-density character of $Y$ and $Y^*$, respectively, whose definition is given below.) Since the condition $\chi(Y) = \chi^*(Y^*)$ holds true for $Y$ reflexive or weakly compactly generated, the above result gives almost immediately a description of complemented subspaces of $\ell_p(\Gamma)$, with $1 < p < \infty$, and $c_0(\Gamma)$ (see [15, 5]), generalizing Pełczyński’s...
classical theorem asserting that every infinite dimensional complemented subspace of $\ell_p$, with $1 \leq p \leq \infty$, (respectively, $c_0$) is isomorphic to $\ell_p$ (respectively, $c_0$) (see [13, Theorem 2.a.3]). One should mention here that in 1966 a similar result for $\ell_1(\Gamma)$ was obtained by Köthe [12].

In the next section we give a characterization of those spaces $X$ where the condition $\chi(Y) = \chi^*(Y^*)$ holds for all subspaces $Y$ of $X$ (Theorem 1); this appears to be equivalent to the non-containment (by $X$) of an isomorphic copy of the space $\ell_1(\mathbb{N}_1)$ or, under the continuum hypothesis, to the weak* sequential compactness of the unit ball of $X^*$ (Proposition 1). The latter equivalence relates to the problem posed in 1977 by Rosenthal [17]: Suppose that the dual unit ball $B_{W^*}$ of a Banach space $W$ is not weak* sequentially compact; can we then conclude that $W$ contains a subspace isomorphic to $\ell_1^\infty$, which was answered in 1978 in the negative by Hagler and Odell [7] (cf. [6, 11]). Moreover, in 1977 Haydon constructed a Banach space $Z$ (of the type $C(K)$) not containing isomorphic copies of $\ell_1(\mathbb{N}_1)$ such that $B_{Z^*}$ is not weak* sequentially compact [9]. This shows that our equivalence does depend on the structure of the given Banach space (for other results concerning the embeddability of $\ell_1(\mathbb{N}_1)$ into Banach spaces see [19] and the references given therein). The characterization given in Theorem 1 allows us to generalize, by (RSG), the cited results of Rodriguez-Salinas and Granero (Theorem 2); it also shows that if $X$ contains no copy of $\ell_1(\mathbb{N}_1)$ then $Y$, containing large (unconditional) basic sets consisting of pairwise disjoint elements, has “big” unconditional structure (see the comment in ([4, p. 396]) on atomic Banach lattices). In Section 3, complementing the previous theorems, we show that every uncountable unconditional basic set $(y_j)_{j \in J}$ in $X$ contains a subset of the same cardinality as $J$ consisting of pairwise disjoint elements provided that $(y_j)_{j \in J}$ has no uncountable subsets of the $\ell_1$-type (Theorem 3).

The restrictive role of $\ell_1(\mathbb{N}_1)$, with $J$ uncountable, in Theorems 1, 2, and 3 explains the following result obtained in 1975 by Troyanski [21]

$$(T) \ Let \ the \ basis \ (x_\gamma)_{\gamma \in \Gamma} \ of \ X \ be \ symmetric. \ If \ X \ has \ a \ subspace \ isomorphic \ to \ \ell_1(\mathbb{N}_1) \ [resp., \ c_0(\mathbb{N}_1)] \ for \ some \ uncountable \ set \ J, \ then \ the \ basis \ is \ equivalent \ to \ the \ natural \ basis \ of \ \ell_1(\Gamma) \ [resp., \ c_0(\Gamma)],$$

and generalized in 1988 by Drewnowski [3] who showed that if the basis in $(T)$ is merely unconditional then it contains “large” subbasas of the $\ell_1$-[resp., $c_0$]-type. Therefore, in the context of Troyanski’s result, the last section is devoted only to the structure of nonseparable subspaces of $X = \ell_1(\Gamma)$, and the basic tool we use in our studies is the notion of $\varepsilon$-disjoint systems. In Theorem 4 we prove the existence, for every $\varepsilon > 0$, of such systems in $X$, which allows one to strengthen the above-cited result of Köthe (Corollary 6) and to give its shorter proof (Corollary 7).

Our terminology and notation is that of [13] and [20]. All subspaces are assumed to be linear and closed. Recall that a family $(x_\gamma)_{\gamma \in \Gamma}$ in $X$ is said to be an (unconditional) basis of $X$ if, for every $x \in X$ there is a unique family of scalars $(t_\gamma)_{\gamma \in \Gamma}$ such that $x = \sum_{\gamma \in \Gamma} t_\gamma x_\gamma$ (unconditional convergence). By $(x^*_\gamma)_{\gamma \in \Gamma}$ we denote the dual family, biorthogonal to $(x_\gamma)_{\gamma \in \Gamma}$; then

$$x = \sum_{\gamma \in \Gamma} x^*_\gamma(x)x_\gamma \quad \text{for every } x \in X,$$  \hspace{1cm} (1)

and the support of $x \in X$ is defined as $\text{supp}(x) := \{\gamma \in \Gamma : x^*_\gamma(x) \neq 0\}$. We say that two elements $u, v \in X$ are disjoint if their supports, $\text{supp}(u)$ and $\text{supp}(v)$, are disjoint subsets of $\Gamma$. From (1) it follows that every element $x^* \in X^*$ has the representation

$$x^* = \sum_{\gamma \in \Gamma} x^*_\gamma(x_\gamma)x^*_\gamma \quad \text{(weak*-convergence)},$$  \hspace{1cm} (2)

which allows one to define the support of $x^*$ as $\text{supp}(x^*) := \{\gamma \in \Gamma : x^*(x_\gamma) \neq 0\}$. The basis $(x_\gamma)_{\gamma \in \Gamma}$ is called symmetric if, for every sequence $(\gamma_n) \in \Gamma$ the basic sequence $(x_{\gamma_n})$ is symmetric in the usual sense ([13, p. 113]). We say that a family $(v_j)_{j \in J}$ is a basic set in $X$ if it is a basis of the closed linear span of this family (denoted by $[v_j]_{j \in J}$). Two basic sets $(u_j)_{j \in J}, (v_j)_{j \in J}$ in a Banach space $W$ are said to be equivalent if the linear operator $G : [u_j]_{j \in J} \rightarrow [v_j]_{j \in J}$ of the form $G(u_j) = v_j$ is an isomorphism.
An isomorphism \( T \) between two Banach spaces \( V \) and \( W \) is said to be an \((1 + \varepsilon)\)-isometry provided that 
\[
\|T\| \|T^{-1}\| \leq 1 + \varepsilon.
\]

A subspace \( W \) of a Banach space \( W \) is said to be complemented \([k\text{-complemented} \text{ for some } k \geq 1, \text{ resp.}]\) in \( W \) if it is the range of a continuous projection \( P \) [with \( \|P\| = k \), resp.] If \( F \) is a nonempty subset of \( \Gamma \), then \( X_f \) denotes the subspace of \( X \) consisting of the elements with supports included in \( F \), and \( P_f \) denotes the continuous projection from \( X \) onto \( X_f \) of the form \( P_f : \chi \mapsto x \cdot \chi \cdot 1_f \), where \( 1_f \) is the characteristic function of \( F \). Notice that if \( X = \ell_p(\Gamma) \), then the spaces \( X_f \) and \( \ell_p(\Gamma) \) are isometric. From (2) it easily follows that for every \( x^* \in X^* \) the element \( x^*_f := \sum_{\gamma \in F} x^*(\gamma) x^*_\gamma \) (weak*-convergence) is well defined, and hence the operator \( P_f \) on \( X^* \) of the form \( P_f(x^*) := x^*_f \) is a continuous projection (in fact, \( P_f = P_f^* \)).

If \( Y \) is a subspace of \( X \) then \( X \) possesses this property. In the theorem below we give a characterization of the class of those every weakly countably determined (in particular, every weakly compactly generated (WCG)) Banach space \( V \) short) if, for every subspace \( Y \) of \( V \), we obtain that
\[
\text{card}(Y) \leq \text{card}(\varnothing \chi^*(\gamma)) = \text{card}(\varnothing \chi^*(\gamma) \gamma \in \Gamma^*).
\]

From (iii) we obtain that
\[
\text{card}(A) \leq \text{card}(\varnothing \chi^*(\gamma)) = \text{card}(\varnothing \chi^*(\gamma) \gamma \in \Gamma^*),
\]

\[\text{(3)}\]
Moreover, under the continuum hypothesis

\[ A = \{ x^* \in X^* : \text{supp}(x^*) \subset A \} \]

\[ x^*(P_{T \setminus A}y) = x^*(y) \]

for every \( y \in Y \) and \( x^* \in F_0 \). It follows that the set \( F_0 \) is total over \( P_A(Y) \); thus the operator \( P_A \) restricted to \( Y \) is injective which, together with (3), gives

\[ \chi(Y) \leq \text{card}(Y) = \text{card}(P_A(Y)) = \text{card}(\Gamma_Y \cap A) \leq \chi^*(Y^*) \]

Finally, \( \chi(Y) = \chi^*(Y^*) \), as claimed.

If \( X \) has W*CP then (i) holds. Assume that \( X \) contains an isomorphic copy of \( \ell_1(J) \) with \( \text{card}(J) = \aleph_1 \).

The remaining part of the proof depends on the observation that if \( W \) is a separable Banach space then for every infinite dimensional subspace \( Y \) of \( W^* \) we have \( \chi^*(Y^*) = \aleph_0 \), which we apply to the space \( W = C[0,1] \) whose dual contains \( Y := \ell_1([0,1]) \).

As a by-product of the equivalence of (i) and (ii) in Theorem 1 we obtain the Troyanski’s result (T) (see Introduction) which immediately gives

**Corollary 1** Let \( X \) be a Banach space with an uncountable symmetric basis \( (x_\gamma)_{\gamma \in \Gamma} \). Then \( X \) has the W*CP if and only if the basis is not equivalent to the standard basis of \( \ell_1(\Gamma) \).

The above Corollary applies to “big” Orlicz spaces \( h_\varphi(\Gamma) \), where \( \varphi \) is an Orlicz function (for exact definition of \( h_\varphi(\Gamma) \) see e.g. [10]), giving that \( h_\varphi(\Gamma) \) has the W*CP if and only if \( \varphi \) is not equivalent to the linear function \( \psi(t) = t \) at 0.

The next theorem is an immediate consequence of Theorem 1 and the result (RSG); it applies to the spaces \( h_\varphi(\Gamma) \), in particular to \( \ell_2(\Gamma) \), \( 1 < p < \infty \), and \( c_0(\Gamma) \) (cf. [5, 15]). It also complements a similar result obtained in [15, Proposition 2] for \( Y \) reflexive.

**Theorem 2** If \( X \) contains no isomorphic copies of \( \ell_1(\aleph_1) \), then every nonseparable subspace \( Y \) of \( X \) contains a set of the cardinality of \( \chi(Y) \) consisting of pairwise disjoint elements of norm one.

The proposition below deals with weak* sequential compactness of the dual unit ball of \( X^* \). The proof of the first implication is a discrete version of the proof given in 1968 by Lozanovskii [14] for a class of Banach lattices (cf. [23, Theorem 4.4]), and is included here for the convenience of the reader who is not familiar with the theory of Banach lattices (one should also note that the original proof works for real Banach lattices).

**Proposition 1** Let \( X \) be a Banach space with an uncountable unconditional basis. Then statement (iii) in Theorem 1 implies that

(iv) the dual unit ball \( B_{X^*} \) of \( X^* \) is weak* sequentially compact.

Moreover, under the continuum hypothesis (CH) statements (i) and (iv) are equivalent.

**Proof** (iii)\( \Rightarrow \) (iv) Let \( (x^*_n) \) be a sequence in \( B_{X^*} \), and put \( V := \{ x^* \in X^* : \text{supp}(x^*) \subset A \} \), where

\[ A = \bigcup_{n=1}^{\infty} \text{supp}(x^*_n) \]

We obviously have \( V = \hat{P}_A(X^*) \), and \( x^*_n \in V \) for all \( n \)’s. We set \( Y := P_A(X) \).

Since \( A \) is countable, the space \( Y \) is separable. It is easy to check that the annihilator \( \hat{Y}^* \) of \( Y \) in \( X^* \) equals \( \hat{P}_{T \setminus A}(X^*) \), and hence \( Y^* \) can be identified with \( \hat{P}_A(X^*)(= V) \). The separability of \( Y \) implies that the ball \( B_{Y^*} \) is \( \sigma(Y^*, Y) \)-sequentially compact, and using the above identification of \( Y^* \) and \( V \), we can find a \( \sigma(X^*, X) \)-convergent subsequence \( (x^*_n) \) of \( (x^*_n) \).

(iv)\( \Rightarrow \) (i) (under CH; cf [23, pp. 78–79]). It is known that condition (iv) implies \( X \) cannot contain isomorphic copies of \( \ell_1(\mathbb{R}) \), where \( \mathbb{R} \) denotes the set of all real numbers (see e.g. [1, p. 226]), and hence, under CH, the space \( X \) cannot contain any copy of \( \ell_1(\aleph_1) \).

From Corollary 1 and Proposition 1 we immediately obtain

**Corollary 2** Let the basis \( (x_\gamma)_{\gamma \in \Gamma} \) of \( X \) be symmetric. Under the continuum hypothesis, the dual unit ball of \( X \) is weak* sequentially compact if and only if the basis is not equivalent to the standard basis of \( \ell_1(\Gamma) \).
3 Uncountable unconditional basic sets in $X$

In the theorem below we show that large unconditional basic sets in $X$ have “nice” structure (the conclusion (i) below was obtained in [15, Proposition 6] under more restrictive assumption); its application is given in three corollaries following it.

**Theorem 3** Let $X$ be a Banach space with an uncountable unconditional basis, and let $(y_j)_{j \in J}$ be an uncountable unconditional normalized basic set in $X$. Then the following alternative holds:

(i) There is a subset $J_0$ of $J$ with $\text{card}(J_0) = \text{card}(J)$ such that the elements of $(y_j)_{j \in J_0}$ are pairwise disjoint.

(ii) For every infinite cardinal number $\alpha_0 < \text{card}(J)$ there exists a subset $J_0$ of $J$ with $\text{card}(J_0) > \alpha_0$ such that $(y_j)_{j \in J_0}$ is equivalent to the unit vector basis of $\ell_1(J_0)$.

In particular, the conclusion of part (i) holds if $(y_j)_{j \in J}$ is equivalent to the unit vector basis of $c_0(J)$ or $\ell_p(J)$, with $1 < p < \infty$.

We would like to comment on the above property (ii) in Theorem 3. One should note that it is impossible, in general, to choose a pairwise disjoint subsequence even from a sequence $(y_n)$ in $X$ equivalent to the unit vector basis of $\ell_1$: it is enough to take any $\gamma_0 \in \Gamma \setminus \bigcup_{n=1}^{\infty} \text{supp}(y_n)$ and consider the sequence $(y'_n)$, equivalent to $(y_n)$, of the form $y'_n = y_n + x_n$, $n = 1, 2, \ldots$. On the other hand, it is known that a Banach space with an unconditional Schauder basis contains a copy of $\ell_1$ if it contains a normalized block basic sequence of the basis equivalent to the unit vector basis of $\ell_1$ (see e.g. [13, Theorem 1.c.9]).

The proof of Theorem 3 depends essentially on Lemma 1 below and it is a modification of the arguments used in the proof of [2, Lemma 3]. To shorten the text we say that a family $(y_j)_{j \in J}$ of non-null elements of a Banach space $W$ is totally non-$\ell_1(\alpha)$, where $\alpha$ is an infinite cardinal number with $\alpha \leq \text{card}(J)$ (TN$\ell_1(\alpha)$, for short) if, for every subset $C$ of $J$ with $\text{card}(C) = \alpha$ there is a family $(t_j)_{j \in C}$ of scalars such that the series $\sum_{j \in C} t_j y_j$ converges unconditionally, but $\sum_{j \in C} |t_j| = \infty$. (For $\alpha = \aleph_0$ this notion coincides with the notion of a totally non-$\ell_1$ family considered by Drewnowski in [2].) If $(y_j)_{j \in J}$ is a basic set in $X$ then it is totally non-$\ell_1(\alpha)$ whenever, for every subset $C$ of $J$ with $\text{card}(C) = \alpha$, the basic set $(y_j)_{j \in C}$ is not equivalent to the standard basis of $\ell_1(C)$. We have that if $\alpha_1 < \alpha_2$, then TN$\ell_1(\alpha_1)$ implies TN$\ell_1(\alpha_2)$; thus, if $(y_j)_{j \in J}$ is totally non-$\ell_1$ then it is TN$\ell_1(\alpha)$ for every infinite $\alpha \leq \text{card}(J)$.

**Lemma 1** Let $X$ be a Banach space with an uncountable unconditional basis, let $\alpha_0$ be an infinite cardinal number, and let $J$ be a set with $\text{card}(J) > \alpha_0$. If, for every cardinal $\alpha$ with $\alpha_0 < \alpha \leq \text{card}(J)$ a family $(y_j)_{j \in J}$ of non-null elements of $X$ is TN$\ell_1(\alpha)$, then there exists a subset $J_0$ of $J$ with $\text{card}(J_0) = \text{card}(J)$ such that the elements of the subfamily $(y_j)_{j \in J_0}$ are pairwise disjoint.

**Proof.** It is an immediate consequence of the following combinatorial fact, the proof of which is similar to the proof of [2, Lemma 2] and therefore omitted:

Let $J$ be an uncountable set, and let $m$ be an infinite cardinal number with $m < \text{card}(J)$. Let $(S_j)_{j \in J}$ be a family of subsets of a set $\Gamma$ such that:

(a) for every $j \in J$ we have $\text{card}(S_j) \leq m$, and

(b) for every $\gamma \in \Gamma$ we have $\text{card}\{j \in J : \gamma \in S_j\} \leq m$.

Then there exists a subset $J_0$ of $J$ with $\text{card}(J_0) = \text{card}(J)$ such that the elements of the family $(S_j)_{j \in J_0}$ are pairwise disjoint. ■

**The proof of Theorem 3.** Assume condition (ii) is false. Then $(y_j)_{j \in J}$ is TN$\ell_1(\alpha)$ for all cardinal numbers $\alpha$ with $\alpha_0 < \alpha \leq \text{card}(J)$. Now we apply Lemma 1. ■
The two below corollaries of Theorem 3 show that uncountable unconditional basic sets in the spaces \( \ell_p(\Gamma) \) and \( c_0(\Gamma) \) contain long symmetric subsets. (One should note here that subspaces with symmetric uncountable bases in Orlicz spaces \( \ell_\varphi(\Gamma) \) were described by Rodríguez-Salinas [16]; see also [10].)

The first corollary is now obvious (the case \( X = c_0(\Gamma) \) and \( X = \ell_p(\Gamma) \), with \( 1 < p < \infty \), was studied in [5] and [15], respectively).

**Corollary 3** Let \( X \) be a Banach space with an uncountable unconditional basis, and let \((y_j)_{j \in J}\) be an uncountable unconditional normalized basic set in \( X \). If \( X \) contains no isomorphic copy of the space \( \ell_1(\mathbb{N}_1) \), then there exists a subset \( J_0 \) of \( J \) with \( \text{card}(J_0) = \text{card}(J) \) such that the elements of \((y_j)_{j \in J_0}\) are pairwise disjoint.

Each of the either cases of Theorem 3 proves the next corollary.

**Corollary 4** Let \((y_j)_{j \in J}\) be an uncountable unconditional normalized basic set in \( \ell_1(\Gamma) \). Then for every infinite cardinal number \( \alpha_0 < \text{card}(J) \) there exists a subset \( J_0 \) of \( J \) with \( \alpha_0 < \text{card}(J_0) \) and such that the basic subset \((y_j)_{j \in J_0}\) is equivalent to the natural symmetric basis of \( \ell_1(\Gamma_0) \).

It is known that every symmetric basic sequence in the sequence space \( \ell_p \) (or \( c_0 \)) is equivalent to the unit vector basis of the given space [13, Remark following Proposition 3.b.5]. From Corollaries 3 and 4 we immediately obtain a similar property for the spaces \( \ell_p(\Gamma) \) and \( c_0(\Gamma) \).

**Corollary 5** Let \( X(\Gamma) \) denote the space \( \ell_p(\Gamma), 1 \leq p < \infty \), or \( c_0(\Gamma) \). Every uncountable, normalized and symmetric basic set \((y_j)_{j \in J}\) in \( X(\Gamma) \) is equivalent to the natural basis of \( X(J) \).

### 4 \( \varepsilon \)-disjoint systems in \( X \)

The main result of this section is motivated by the remark following Theorem 3 (see also the proof of Theorem 1 in [3]). Here we show that the structure of infinite dimensional subspaces of \( X \) can also be studied effectively by the use of “almost” disjoint elements.

Let \( \varepsilon \in (0, 1) \), and let \( Y \) be a subspace of \( X \). We say that two elements \( y_1, y_2 \in X \setminus \{0\} \) are \( \varepsilon \)-disjoint if there exist disjoint elements \( u_1, u_2 \in X \setminus \{0\} \) such that \( \|x_i - u_i\| \leq \varepsilon, i = 1, 2 \). A system \((y_j)_{j \in J} \subset S_Y \) is said to be \( \varepsilon \)-disjoint provided that there exists a system \((u_j)_{j \in J} \) of pairwise disjoint elements of \( X \) with \( \|y_j - u_j\| \leq \varepsilon \) for all \( j \in J \). A concrete \( \varepsilon \)-disjoint system \((y_j)_{j \in J} \) with the corresponding pairwise disjoint system \((u_j)_{j \in J} \) will be denoted by \((y_j, u_j)_{j \in J}\).

**Remark 1** It is obvious that every \( 0 \)-disjoint system is pairwise disjoint.

**Remark 2** From inequality \( \min\{|a| + |b|, |c|\} \leq \min\{|a|, |c|\} + \min\{|b|, |c|\}, \) for all scalars \( a, b, c \) (see [18, Corollary, p. 53]), we easily obtain that \( \min\{|a|, |b|\} \leq 2|a - u| + |b - v| + \min\{|u|, |v|\} \) for all \( a, b, u, v \). Hence, if the elements \( y_i = \sum_{\gamma \in \Gamma} \ell_1(\gamma) x_{\gamma}, i = 1, 2 \), are \( \varepsilon \)-disjoint, with corresponding disjoint elements \( u_i = \sum_{\gamma \in \Gamma} s_1(\gamma) x_{\gamma}, i = 1, 2 \), then \( \|\sum_{\gamma \in \Gamma} \min\{|t_\gamma|, |s_\gamma|\} x_{\gamma}\| \leq 2K\|y_1 - u_1\| + K\|y_2 - u_2\| \leq 3K\varepsilon, \) where \( K \) is the basis constant. It follows that the supports of two \( \varepsilon \)-disjoint elements intersect at “norm-min-small” subsets.

**Remark 3** Let \( \varepsilon \in (0, 1/2) \), and let a system \((y_j, u_j)_{j \in J} \) be \( \varepsilon \)-disjoint in \( \ell_1(\Gamma) \). Then \((y_j)_{j \in J} \) is equivalent to the standard basis of \( \ell_1(J) \), with \( \|u_j\| \in (1 - \varepsilon, 1 + \varepsilon) \) for all \( j \)’s, and similarly for \((y_j)_{j \in J} \):

\[
\sum_{j \in J} |t_j| \geq \|\sum_{j \in J} t_j y_j\| \geq (1 - 2\varepsilon) \sum_{j \in J} |t_j|,
\]

for all \((t_j)_{j \in J} \in \ell_1(J) \). Proving as in [13, Proposition 1.a.9 and Theorem 2.a.3], we obtain that for \( \varepsilon \in (0, \sqrt{2} - 1) \), the spaces \([y_j]_{j \in J} \) and \([u_j]_{j \in J} \) are \((1 + \varepsilon)\)-isometric and \( \delta \)-complemented in \( \ell_1(\Gamma) \), where \( \delta \leq 1 + \frac{2\varepsilon}{1 - 2\varepsilon - \varepsilon^2} \).
The main result of this section reads as follows.

**Theorem 4** Let $Y$ be an infinite dimensional subspace of $X$. Then for every $\varepsilon \in (0, 1)$ the space $Y$ contains an $\varepsilon$-disjoint system $(y_j, u_j)_{j \in J}$ with $\text{card}(J) = \chi(Y)$ and such that $\text{supp}(u_j) \subset \text{supp}(y_j)$ for all $j \in J$.

**Proof.** Put $F = \text{supp}(Y)$. We first consider the case $\chi(Y) = \aleph_0$. Then $F$ is countable, and hence $Y$ is a subspace of the space $X_F$ with the countable unconditional basis $(x_\gamma)_{\gamma \in F}$. By [13, Proposition 1.a.11], $Y$ contains an $\varepsilon$-disjoint countable infinite system $(y_n, u_n)_{n \geq 1}$ with $\text{supp}(u_n) \subset \text{supp}(y_n)$ for all $n$’s.

Now assume $\chi(Y) > \aleph_0$, and let $\mathcal{E}$ be the class of all $\varepsilon$-disjoint systems $(y_j, v_j)_{j \in J}$ with $\text{card}(J) \geq \aleph_0$ and $\text{supp}(u_j) \subset \text{supp}(y_j)$ for all $j \in J$. By the previous case, $\mathcal{E} \neq \emptyset$. We introduce the following partial ordering in $\mathcal{E}$: $(y_j', v_j')_{j \in J} \preceq (y_j'', v_j'')_{j \in J}$ iff $J \subset L$ and $y_j' = y_j''$ and $v_j' = v_j''$ for all $j \in J$, and let $(y_j^M, v_j^M)_{j \in J_M}$ be a maximal element in $\mathcal{E}$. We define the cardinal number $\lambda_M := \text{card}(J_M)$, and we put $I_M := \bigcup_{j \in J_M} \text{supp}(u_j^M)$. Then we have

$$\lambda_M = \text{card}(I_M) \leq \chi(Y).$$

We claim we have two equalities in (4). Assume this is not so, i.e., $\lambda_M < \chi(Y)$. Then we must have:

$$\text{for every } \eta > 0 \text{ there is } y_\eta \in S_Y \text{ with } \|P_{I_M}y_\eta\| < \eta \quad (*)$$

(in the opposite case the number $\inf_{y \in S_Y} \|P_{I_M}y\|$ were positive, and hence the operator $P_{I_M}$ restricted to $Y$ would be injective; this and (4) would then imply that $\chi(Y) \leq \text{card}(Y) = \text{card}(P_{I_M}(Y)) = \text{card}(F \cap I_M) \leq \lambda_M < \chi(Y)$, a contradiction). Now choose $y_\eta \in S_Y$ fulfilling (*) with $\eta = \varepsilon$, and put $w_\eta = P_{I_M}y_\eta$; we see that $\text{supp}(w_\eta) \subset \text{supp}(y_\eta)$. Next, from (*) we obtain $\|y_\eta - w_\eta\| < \varepsilon$, and since $\text{supp}(w_\eta) \cap I_M = \emptyset$, we also have that for every $j \in J_M$ the elements $u_\eta$ and $u_j$ are disjoint. It follows that for the set $J^\eta := J_M \cup \{\eta\}$ the system $(y_j, u_j)_{j \in J^\eta}$ is $\varepsilon$-disjoint, and it strictly dominates $(y_j, u_j)_{j \in J}$. This contradiction proves our claim and finishes the proof. □

From Theorem 4 and Remark 3 we get

**Corollary 6** Let $Y$ be a nonseparable subspace of $\ell_1(\Gamma)$. Then for every $\varepsilon \in (0, \sqrt{2} - 1)$ the space $Y$ contains an $(1 + \varepsilon)$-isometric and $\delta$-complemented copy of $\ell_1(J)$, where $\text{card}(J) = \chi(Y)$ and $\delta \leq 1 + \frac{2\varepsilon}{\sqrt{2} - 2 - \varepsilon^2}$.

Consequently, from Corollary 6 and Pelczyński’s decomposition method we obtain Köthe’s result ([12, Theorem (6), p. 187]):

**Corollary 7** Every complemented subspace of $\ell_1(\Gamma)$ is isomorphic to $\ell_1(J)$ for some $J \subset \Gamma$.

**References**


