The Peano curves as limit of $\alpha$-dense curves

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Abstract. In this paper we present a characterization of the Peano curves as the uniform limit of sequences of $\alpha$-dense curves contained in the compact that it is filled by the Peano curve. These $\alpha$-dense curves must have densities tending to zero and coordinate functions with variation tending to infinite as $\alpha$ tends to zero.

Las curvas de Peano como límite de curvas $\alpha$-densas

Resumen. En este artículo presentamos una caracterización de las curvas de Peano como límite uniforme de sucesiones de curvas $\alpha$-densas en el compacto que es llenado por la curva de Peano. Estas curvas $\alpha$-densas deben tener densidades tendiendo a cero y sus funciones coordenadas deben de ser de variación tendiendo a infinito cuando $\alpha$ tiende a cero.

1 Introduction

In a metric space $(E, d)$, given a compact set $K$ and a real number $\alpha \geq 0$, an $\alpha$-dense curve (more information on these curves may be found in [4]) in $K$ is a continuous mapping $\gamma_\alpha : I \to E$, with $I = [0, 1]$, satisfying

i) the image $\gamma_\alpha(I)$, from now on noted $\gamma_\alpha^*$, is contained in $K$,

ii) for any $x \in K$, the distance $d(x, \gamma_\alpha^*) \leq \alpha$.

Whenever $\alpha = 0$, one has a Peano curve provided that the interior of $K$ to be non-void. The minimal $\alpha$ verifying the two preceding properties is, strictly speaking, the density of the curve in $K$, which coincides with the Hausdorff distance $d_H(K, \gamma_\alpha^*)$ (see [2]).

A compact subset $K$ in $(E, d)$ is said to be densifiable if it contains $\alpha$-dense curves for arbitrary $\alpha > 0$. For example, in $\mathbb{R}^N$, $N \geq 1$, any cube $\prod_{i=1}^{N} [a_i, b_i]$ is densifiable. Any Peano Continuum, that is, a connected and locally connected compact set, is also densifiable. However, there exist densifiable sets which are not Peano Continua; for instance

$$K = \left\{ \left( x, \sin \frac{1}{x} \right) : 0 < x \leq 1 \right\} \cup \left\{ (0, y) : -1 \leq y \leq 1 \right\}.$$ 

Therefore, the $\alpha$-density concept produces a new class, the densifiable sets, which is strictly between the class of Peano Continua and the class of connected and precompact sets.
Let $f$ be a function defined on a real interval, for brevity we take the unit interval $I$, and valued on a metric space $(E, d)$. We recall that the total variation of $f$, noted $V_I(f)$, is defined as

$$V_I(f) \equiv \sup_{\sigma} \left\{ \sum_{i=1}^{n} d(f(t_i), f(t_{i-1})) : \sigma \equiv \{t_0, t_1, \ldots, t_n\} \subset I ; t_0 < t_1 < \cdots < t_n \right\}.$$  

Whenever $V_I(f) < \infty$, it is well-known that $f$ is called of bounded variation on $I$ (detailed properties of these functions can be found, for instance, in [1] or also in [6, Vol. I]). In particular, given a continuous mapping $\gamma : I \rightarrow \mathbb{R}^n$, i.e., a curve $\gamma$, the total variation $V_I(\gamma)$ is also called the length, written $L(\gamma)$. Whether $V_I(\gamma)$ is finite, the curve is said to be rectifiable and its length may be determined (see [1, theorem 24-6]) by

$$L(\gamma) = \lim_{|\Pi| \to 0} \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|,$$

$\Pi$ being the partition $\Pi = \{t_0, t_1, \ldots, t_n\}$; $0 = t_0 < t_1 < \cdots < t_n = 1$

with norm

$$|\Pi| \equiv \max \{t_i - t_{i-1} : i = 1, \ldots, n\}.$$

The variation of a curve may be infinite even for very regular one, such as the following example shows (see [8, p. 53]).

**Example 1** The coordinate functions $\gamma_1$, $\gamma_2$ of the spiral $\gamma = (\gamma_1, \gamma_2) : I \rightarrow I^2$ defined by

$$\gamma_1(t) = \begin{cases} \frac{t \cos \frac{2\pi}{r}}{r} & \text{if } 0 < t \leq 1 \\ 0 & \text{if } t = 0 \end{cases} \quad \gamma_2(t) = \begin{cases} \frac{t \sin \frac{2\pi}{r}}{r} & \text{if } 0 < t \leq 1 \\ 0 & \text{if } t = 0 \end{cases}$$

are both of infinite variation.

## 2 The theorem of characterization

The Hahn-Mazurkiewicz theorem (see [7]) assures that every Peano Continuum set in a metrizable space is the continuous image of the unit interval, and reciprocally. Since the unit square $I^2$ is a Peano Continuum, it may be taken as a good prototype of the image of a Peano curve, so we shall state our theorem of characterization in that set.

**Theorem 1** A continuous mapping $\gamma = (\gamma_1, \gamma_2) : I \rightarrow I^2$ is a Peano curve filling $I^2$ if and only if is the uniform limit of a sequence of $\alpha$-dense curves $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$ in $I^2$ with densities $\alpha_n \to 0$, for which there is no constant $K$ such that the variation $V_I(\gamma_i^{(n)}) \leq K$, for all $n$, for some $i = 1, 2$.

**Proof.** First we prove the sufficiency. Let $P$ be an arbitrary point of $I^2$; because of the density, for each $n$ there exists $t_n \in I$ such that the euclidean distance

$$d(P, \gamma^{(n)}(t_n)) \leq \alpha_n.$$

By the Bolzano-Weierstrass theorem, given the sequence $(t_n)_n$ there exists a subsequence, noted in the same way, that converges to some $t \in I$. For arbitrary $n$, we consider the inequality

$$d(P, \gamma(t)) \leq d(P, \gamma^{(n)}(t_n)) + d(\gamma^{(n)}(t_n), \gamma^{(n)}(t)) + d(\gamma^{(n)}(t), \gamma(t)). \quad (1)$$

Thus, since $\alpha_n \to 0$ and $\gamma$ is the uniform limit of $\gamma_n$, from the continuity of the curves and taking the limit in (1) when $n \to \infty$, the distance $d(P, \gamma(t)) = 0$. Therefore, the point $P = \gamma(t)$ and so $\gamma$ is a Peano curve that fills $I^2$.  

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For proving the necessity, observe that if $\gamma = (\gamma_1, \gamma_2)$ is a Peano curve filling $I^2$, then each coordinate function $\gamma_1, \gamma_2$ is necessarily surjective onto $I$. We assume firstly that $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$, $n = 1, 2, \ldots$, is a sequence of curves in $I^2$ uniformly convergent

$$\lim_{n \to \infty} \gamma^{(n)} = \gamma,$$  \hspace{1cm} (2)

and prove that latter. □

Denoting by $\alpha_n$ the density in $I^2$ of each curve $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$, one has

$$\lim_{n \to \infty} \alpha_n = 0.$$  \hspace{1cm} (3)

Indeed, if (3) is not true, then there exists $\epsilon > 0$ such that for any $k$ there is an integer $N_k$ so that $\alpha_{N_k} > \epsilon$. Thus we can select a subsequence of curves of densities $\alpha_{N_k} > \epsilon$ for $k = 1, 2, 3, \ldots$. From (2) the limit of this subsequence is also $\gamma$, so denoting the subsequence in the same way, we determine, for each $n$, a point $P_n$ such that

$$\epsilon < d(P_n, \gamma_n^*) \leq \alpha_n.$$  \hspace{1cm} (4)

Since $(P_n)_n$ belongs to the compact $I^2$, there exists a subsequence, noted in the same way, that converges to some point $P \in I^2$. Because of the continuity of the distance function, and taking into account that $\gamma$ is the uniform limit of $\gamma_n$, given $0 < \delta < \epsilon$, there exists a sufficiently large $n$ such that

$$|d(P, \gamma_n^*) - d(P_n, \gamma_n^*)| < \frac{\delta}{2}; \quad |d(P, \gamma^*) - d(P_n, \gamma_n^*)| < \frac{\delta}{2}.$$  \hspace{1cm} (5)

From (5) and (4), one has

$$d(P, \gamma^*) = d(P, \gamma^*) - d(P, \gamma_n^*) + d(P, \gamma_n^*) - d(P_n, \gamma_n^*) + d(P_n, \gamma_n^*) > -\frac{\delta}{2} - \frac{\delta}{2} + d(P_n, \gamma_n^*) > \epsilon - \delta,$$

which is absurd because $d(P, \gamma^*) = 0$. Therefore (3) is showed.

For each $i = 1, 2$, consider the Banach indicatrix $\Phi_{\gamma_i}$ of each coordinate function $\gamma_i$ on the interval $[0, 1]$, that is, the function on $I$ defined by

$$\Phi_{\gamma_i}(y) = \begin{cases} +\infty & \text{if card}(\gamma_i^{-1}(y)) \geq \omega \\ \text{card}(\gamma_i^{-1}(y)) & \text{if card}(\gamma_i^{-1}(y)) < \omega \end{cases}$$

$\omega$ being the first infinite cardinal. $\Phi_{\gamma_i}$ is measurable and satisfies the integral formula

$$\int_{0}^{1} \Phi_{\gamma_i}(y)dy = V_I(\gamma_i)$$  \hspace{1cm} (6)

(a proof can be found in [3] or [6]). Nevertheless $\Phi_{\gamma_i}$ is identically equal to $+\infty$ on $I$, so from (6)

$$V_I(\gamma_i) = \infty, \quad i = 1, 2.$$  \hspace{1cm} (7)

Suppose the existence of a constant $K$ such that $V_I(\gamma_i^{(n)}) \leq K$, for all $n$, for some $i = 1, 2$. Thus, as $0 \leq \gamma_i^{(n)}(t) \leq 1$ for any $t \in I$, by applying the Helly’s first theorem (see [6, Vol. I, p.222]), $\gamma_i$ would be of finite variation and it contradicts (7).

Now, it only remains to prove that, given a Peano curve $\gamma = (\gamma_1, \gamma_2)$ filling $I^2$ there exists a sequence $\gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)})$, $n = 1, 2, \ldots$, of curves in $I^2$ verifying (2). For that, consider the class $\mathcal{C}$ of all rectangles $C = J_1 \times J_2$ of $I^2$, where $J_1, J_2$ are intervals contained in $I$, and define on this class the set function $\mu$ by

$$\mu(C) = \Lambda_1[\gamma_1^{-1}(J_1) \cap \gamma_2^{-1}(J_2)],$$  \hspace{1cm} (8)
Furthermore, inductively, given the partition and disjoint subsquares of instance its center, noted \( P \) consists ofous functions, say coincides with \( \gamma \in I \) a partition of \( C \)

One can easily check that formula (8) defines a Borel measure on the unit square, which will be also denoted \( \mu \). This measure, associated to the Peano curve \( \gamma \), satisfies

a) \( \mu(C) > 0 \) for any rectangle \( C \) with interior non-void,

b) \( \mu(I^2) = 1 \).

Now, for each \( n = 1, 2, \ldots \) consider a partition \( \Pi_n = \left\{ C_k^{(n)} : k = 1, 2, \ldots, 2^{2n} \right\} \) formed by \( 2^{2n} \) equal and disjoint subsquares of \( I^2 \), arranged in such a way that \( C_k^{(n)} \) to be adjacent to \( C_{k-1}^{(n)} \) for \( k = 2, \ldots, 2^{2n} \). Furthermore, inductively, given the partition \( \Pi_n \), the next one \( \Pi_{n+1} = \left\{ C_k^{(n+1)} : k = 1, 2, \ldots, 2^{2(n+1)} \right\} \), obtained by dividing each square \( C_k^{(n)} \) into four new squares \( C_{k,i}^{(n)} \), \( i = 1, \ldots, 4 \), is arranged by defining

\[
C_{k(i-1)+1}^{(n+1)} = C_{k,i}^{(n)}, \quad k = 1, 2, \ldots, 2^n, \quad i = 1, \ldots, 4.
\]

From the properties a), b), the \( 2^{2n} \) subintervals

\[
I_{1}^{(n)} = \left[ 0, \mu(C_1^{(n)}) \right], \\
I_{2}^{(n)} = \left[ \mu(C_1^{(n)}), \mu(C_1^{(n)}) + \mu(C_2^{(n)}) \right], \\
\vdots \\
I_{2^n}^{(n)} = \left[ \mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \mu(C_{2^{2n-1}}^{(n)}) + 1 \right]
\]

define a partition of \( I \).

Given \( n \), for each \( k = 1, 2, \ldots, 2^n \), we distinguish an arbitrary interior point of each square \( C_k^{(n)} \), for instance its center, noted \( P_k^{(n)} = (x_k^{(n)}, y_k^{(n)}) \), and define on \( I \) the functions

\[
h_1^{(n)}(t) = x_k^{(n)}, \quad t \in I_k^{(n)}, \\
h_2^{(n)}(t) = y_k^{(n)}, \quad t \in I_k^{(n)}.
\]

Observe that, for each \( n \), \( h_1^{(n)} \), \( h_2^{(n)} \) are, possibly, discontinuous at the points \( t_j = \sum_{i=1}^{j} \mu(C_i^{(n)}) \), \( j = 1, 2, \ldots, 2^n - 1 \). However, the sequences \( \left( h_1^{(n)} \right)_n \), \( \left( h_2^{(n)} \right)_n \) are uniformly convergent to two continuous functions, say \( \gamma_1', \gamma_2' \), respectively (consult [5]). Therefore one defines a curve \( \gamma' = (\gamma_1', \gamma_2') \) which coincides with \( \gamma = (\gamma_1, \gamma_2) \), if we take into account that, for each \( n \), the mapping \( \gamma'(n)(t) = (h_1^{(n)}(t), h_2^{(n)}(t)) \), \( t \in I \), coincide with \( \gamma(t) = (\gamma_1(t), \gamma_2(t)) \), \( t \in I \), at least at \( 2^{2n} \) values for \( t \), corresponding to the \( 2^{2n} \) centers of the subsquares \( C_k^{(n)} \) of the partition \( \Pi_n \).

To eliminate the discontinuity of \( h_1^{(n)}, h_2^{(n)} \), we proceed to make a linear interpolation. Hence, consider a partition of \( I \) formed by the subintervals

\[
J_1^{(n)} = \left[ 0, \frac{2^{2n} - 1}{2^n} \mu(C_1^{(n)}) \right], \\
K_1^{(n)} = \left[ \frac{2^{2n} - 1}{2^n} \mu(C_1^{(n)}), \mu(C_1^{(n)}) + \frac{1}{2^n} \mu(C_1^{(n)}) \right].
\]
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\[ J_2^{(n)} = \left[ \mu(C_1^{(n)}) + \frac{1}{2^{2n}} \mu(C_2^{(n)}) + \mu(C_3^{(n)}) - \frac{2^{2n} - 1}{2^{2n}} \mu(C_2^{(n)}) \right], \]

\[ K_2^{(n)} = \left[ \mu(C_1^{(n)}) + \frac{2^{2n} - 1}{2^{2n}} \mu(C_2^{(n)}) + \mu(C_3^{(n)}) + \frac{1}{2^{2n}} \mu(C_3^{(n)}) \right], \]

\[ : \]

\[ K_{2^{2n-1}}^{(n)} = \left[ \mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \frac{2^{2n} - 1}{2^{2n}} \mu(C_{2^{2n}-1}^{(n)}), \right. \]

\[ \mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \mu(C_{2^{2n}-1}^{(n)}) + \frac{1}{2^{2n}} \mu(C_{2^{2n}}^{(n)}) \right]. \]

\[ J_{2^{2n}}^{(n)} = \left[ \mu(C_1^{(n)}) + \mu(C_2^{(n)}) + \cdots + \mu(C_{2^{2n}-1}^{(n)}) + \frac{1}{2^{2n}} \mu(C_{2^{2n}}^{(n)}) \right], \]

and define, for each \( n \), the new functions \( f_1^{(n)}, f_2^{(n)} \) by

\[ f_1^{(n)}(t) = \begin{cases} 
    h_1^{(n)}(t) & \text{if } t \in J_k^{(n)}, k = 1, 2, \ldots, 2^n, \\
    x_j^{(n)} + \frac{x_{j+1}^{(n)} - x_j^{(n)}}{s_j^{(n)} - r_j^{(n)}}(t - r_j^{(n)}) & \text{if } t \in K_j^{(n)}, j = 1, 2, \ldots, 2^n - 1
\end{cases} \]

and

\[ f_2^{(n)}(t) = \begin{cases} 
    h_2^{(n)}(t) & \text{if } t \in J_k^{(n)}, k = 1, 2, \ldots, 2^n, \\
    y_j^{(n)} + \frac{y_{j+1}^{(n)} - y_j^{(n)}}{s_j^{(n)} - r_j^{(n)}}(t - r_j^{(n)}) & \text{if } t \in K_j^{(n)}, j = 1, 2, \ldots, 2^n - 1
\end{cases} \]

where \( r_j^{(n)}, s_j^{(n)} \) are the end-points of \( K_j^{(n)} \).

From the uniform convergence of \( (h_1^{(n)})_n, (h_2^{(n)})_n \) to \( \gamma_1, \gamma_2 \), it follows easily that the sequences \( (f_1^{(n)})_n, (f_2^{(n)})_n \) also converge uniformly to \( \gamma_1, \gamma_2 \), respectively, if we take into account that \( J_k^{(n)} \subset I_k^{(n)} \), for all \( k = 1, 2, \ldots, 2^n \), and \( K_j^{(n)} \) is a closed neighbourhood of \( t_j \) of length \( 1 \frac{1}{2^{2n}} \mu(C_j^{(n)}) + \mu(C_{j+1}^{(n)}) \) for all \( j = 1, 2, \ldots, 2^n - 1 \). Therefore, by defining, for each \( n \), \( \gamma^{(n)} = (f_1^{(n)}, f_2^{(n)}) \) we have definitely a sequence of curves satisfying (2). Now the proof is complete.

Suppose we apply this last theorem, thus the following is immediate.

**Corollary 1** Let \( \gamma^{(n)} = (\gamma_1^{(n)}, \gamma_2^{(n)}) \) be an arbitrary sequence of cartesian (for all \( n \) is \( \gamma_1^{(n)} = I_d \), the identity) \( \alpha \)-dense curves in \( I^2 \) with densities \( \alpha_n \to 0 \). Thus \( (\gamma^{(n)})_n \) has no uniform limit.

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**References**


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