A Sharp Estimate for Bilinear Littlewood-Paley Operator

Lanzhe Liu

Abstract. We establish a sharp estimate for bilinear Littlewood-Paley operator. As application, we obtain the weighted norm inequalities and $L \log L$ type estimate for the bilinear operator

1 Introduction

It is well known that the singular integral operators and their commutators are of importance in many applications (see [5, 9, 16]). As the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [2–7, 12–15]. Let $T$ be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rochberg and Weiss (see [5]) states that the commutator $[b, T](f) = T(bf) − bT(f)$ (where $b \in BMO(\mathbb{R}^n)$) is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. In [2–4], Cohen and Gosselin study the $L^p(\mathbb{R}^n)$ boundedness of the multilinear singular integral operator $T^A$ defined by

$$T^A(f)(x) = \int_{\mathbb{R}^n} \frac{R_{m+1}(A; x, y)}{|x−y|^m} K(x, y)f(y) \, dy.$$ 

However, it has known that the commutator $[b, T]$ is not bounded, in general, from $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$. In [13], C. Pérez proves that the commutator $[b, T]$ satisfies a $L \log L$ type estimate. In [10], Hu and Yang obtain a variant sharp estimate for the multilinear singular integral operators. The main purpose of this paper is to establish a sharp estimate for the bilinear operator associated to the Littlewood-Paley operator and $BMO(\mathbb{R}^n)$ function.

2 Preliminaries and Theorems

In this paper, we will study a class of bilinear operators related to Littlewood-Paley operators, whose definitions are the following.

Let $\psi$ be a function on $\mathbb{R}^n$ which satisfies the following properties:
1. $\int \psi(x) \, dx = 0$;
2. $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$;
3. $|\psi(x + y) - \psi(x)| \leq C |y| (1 + |x|)^{-(n+2)}$ when $2|y| < |x|$. 

Let $m$ be a positive integer and let $A$ be a function on $\mathbb{R}^n$. The bilinear Littlewood-Paley operator is defined by

$$
g_{\psi}^A(f)(x) = \left( \int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2},
$$

where

$$
F_t^A(f)(x) = \int_{\mathbb{R}^n} f(y) \frac{\psi_t(x-y)}{|x-y|^m} R_{m+1}(A; x, y) \, dy,
$$

and

$$
R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D_\alpha A(y)(x-y)^\alpha
$$

and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. Set $F_t(f)(x) = f * \psi_t(x)$. We also define that

$$
g_{\psi}(f)(x) = \left( \int_0^\infty |f * \psi_t(x)|^2 \frac{dt}{t} \right)^{1/2},
$$

which is the Littlewood-Paley operator (see [16]).

Let $H$ be the Hilbert space $H = \{ h : \|h\| = \left( \int_0^\infty |h(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \}$. Then for each fixed $x \in \mathbb{R}^n$, $F_t^A(f)(x)$ and $F_t(f)(x)$ may be viewed as a mapping from $(0, +\infty)$ to $H$, and it is clear that

$$
g_{\psi}^A(f)(x) = \|F_t^A(f)(x)\|, \quad g_{\psi}(f)(x) = \|F_t(f)(x)\|.
$$

Note that when $m = 0$, $g_{\psi}^A$ is just the commutator of the Littlewood-Paley operator (see [11]). While when $m > 0$, it is non-trivial generalizations of the commutators. It is well known that the Littlewood-Paley operator, as the vector-valued singular integral operators, is of great interest in harmonic analysis (see [15]). The purpose of this paper is to establish a sharp estimate for the bilinear operator, then the weighted norm inequalities and $L \log L$ type estimate for the bilinear operator are obtained by using the sharp estimate. We point out that some of our ideas in this paper come from the paper [1] of Álvarez and Pérez.

First, let us introduce some notation (see [8, 9, 13]).

For any locally integrable function $f$, the sharp function of $f$ is defined by

$$
f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,
$$

where, and in what follows, $Q$ will denote a cube with sides parallel to the axes, and $f_Q = |Q|^{-1} \int_Q f(x) \, dx$. It is well-known that

$$
f^\#(x) = \sup_{x \in Q \subset C} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy.
$$

We say that $f$ belongs to $BMO(\mathbb{R}^n)$ if $f^\#$ belongs to $L^\infty(\mathbb{R}^n)$. For $0 < r < \infty$, we denote $f^{\#r}$ by

$$
f^{\#r}_r(x) = \left( |f^r|^{\#}(x) \right)^{1/r}.
$$

Let $M$ be the Hardy-Littlewood maximal operator, that is

$$
M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| \, dy.
$$
We write that $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. For $k \in \mathbb{N}$, we denote by $M^k$ the operator $M$ iterated $k$ times, i.e., $M^k f(x) = M(M^{k-1} f)(x)$ and

$$M^k(f)(x) = M(M^{k-1}(f))(x) \quad \text{when } k \geq 2.$$ 

Let $B$ be a Young function and $\tilde{B}$ be the complementary associated to $B$. Set, for a function $f$,

$$\|f\|_{B,Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q B \left( \frac{|f(y)|}{\lambda} \right) \, dy \leq 1 \right\}.$$ 

The maximal function associated to $\|f\|_{B,Q}$ is defined by

$$M_B(f)(x) = \sup_{x \in Q} \|f\|_{B,Q}.$$ 

The main Young function to be using in this paper is $B(t) = t(1 + \log^+ t)$ and its complementary $\tilde{B}(t) \approx \exp t$, the corresponding maximal functions denoted by $M_{L \log L}$ and $M_{\exp L}$. From [13], we have the generalized Hölder’s inequality

$$\frac{1}{|Q|} \int_Q |f(y)g(y)| \, dy \leq \|f\|_{B,Q} \|g\|_{B,Q}$$

and the following equivalence, for any $x \in \mathbb{R}^n$,

$$M_{L \log L}(f)(x) \approx CM^2(f)(x).$$

From the John-Nirenberg inequality (see [9]), we have the following inequalities, for all cube $Q$ and any $b \in \text{BMO}(\mathbb{R}^n)$,

$$\|b - b_Q\|_{\exp L, Q} \leq C\|b\|_{\text{BMO}}$$

and

$$|b_{2^{k+1}Q} - b_{2Q}| \leq 2^k\|b\|_{\text{BMO}}.$$ 

We denote the Muckenhoupt weights by $A_p$ for $1 \leq p < \infty$ (see [9]).

Now we are in position to state our results.

**Theorem 1** Let $D^\alpha A \in \text{BMO}(\mathbb{R}^n)$, $|\alpha| = m$. Then for any $0 < r < 1$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$(g_\psi^A(f))^\#(x) \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} M^2(f)(x).$$

**Theorem 2** Let $1 < p < \infty$ and $D^\alpha A \in \text{BMO}(\mathbb{R}^n)$, $|\alpha| = m$, $\omega \in A_p$. Then $g_\psi^A$ is bounded on $L^p(\omega)$, that is

$$\|g_\psi^A(f)\|_{L^p(\omega)} \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \|f\|_{L^p(\omega)}.$$

**Theorem 3** Let $D^\alpha A \in \text{BMO}(\mathbb{R}^n)$ for $|\alpha| = m$ and $w \in A_1$. Then there exists a constant $C > 0$ such that for each $\lambda > 0$,

$$w(\{x \in \mathbb{R}^n : g_\psi^A(f)(x) > \lambda\}) \leq C \Phi \left( \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \right) \int_{\mathbb{R}^n} \frac{|f(x)|}{\lambda} \left( 1 + \log^+ \left( \frac{|f(x)|}{\lambda} \right) \right) w(x) \, dx,$$

where $\Phi(t) = t(1 + \log^+ t)$.

As in [13, 15], Theorem 2 and 3 follow from Theorem 1. So we only need to prove Theorem 1.
3 Some lemmas

We begin with some preliminary lemmas.

**Lemma 1 (Kolmogorov, [9, p. 485])** Let $0 < p < q < \infty$ and for any function $f \geq 0$. Set

$$
\|f\|_{WL^q} = \sup_{\lambda > 0} \lambda \left| \{ x \in \mathbb{R}^n : f(x) > \lambda \} \right|^{1/q},
$$

$$
N_{p,q}(f) = \sup_E \frac{\|f\chi_E\|_{L^p}}{\|\chi_E\|_{L^r}}, \quad (1/r = 1/p - 1/q),
$$

where the sup is taken for all measurable sets $E$ with $0 < |E| < \infty$. Then

$$
\|f\|_{WL^q} \leq N_{p,q}(f) \leq \left( \frac{q}{q-p} \right)^{1/p} \|f\|_{WL^p}.
$$

**Lemma 2 ([2, p. 448])** Let $A$ be a function on $\mathbb{R}^n$ and $D^\alpha A \in L^q(\mathbb{R}^n)$ for all $\alpha$ with $|\alpha| = m$ and some $q > n$. Then

$$
|R_m(A;x,y)| \leq C|x-y|^m \sum_{|\alpha| = m} \left( \frac{1}{|Q(x,y)|} \left| \int_{Q(x,y)} |D^\alpha A(z)|^q \, dz \right| \right)^{1/q},
$$

where $Q(x,y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x-y|$.

**Lemma 3 ([13, p. 165])** Let $w \in A_1$. Then there exists a constant $C > 0$ such that for any function $f$ and for all $\lambda > 0$, $w(\{y \in \mathbb{R}^n : M^2(f)(y) > \lambda \}) \leq C \lambda^{-1} \int_{\mathbb{R}^n} |f(y)| (1 + \log^+ (\lambda^{-1} |f(y)|)) \, w(y) \, dy.$

**Lemma 4** Let $1 < p < \infty$ and $D^\alpha A \in BMO(\mathbb{R}^n)$ for $|\alpha| = m$, $1 < r \leq \infty$, $1/q = 1/p + 1/r$. Then $g^A_\psi$ is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, that is

$$
\|g^A_\psi(f)\|_{L^q} \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{L^r} \|f\|_{L^p}.
$$

**Proof.** By Minkowski inequality and the condition of $\psi$, we have

$$
g^A_\psi(f)(x) \leq \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^m} \left( \int_0^\infty |\psi'(s)|^2 \frac{ds}{t} \right)^{1/2} \, dy
$$

$$
\leq C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^m} \left( \int_0^\infty \frac{t^{-2n}}{(1+|x-y|/t)^{2(n+1)}} \frac{dt}{t} \right)^{1/2} \, dy
$$

$$
\leq C \int_{\mathbb{R}^n} \frac{|R_{m+1}(A;x,y)|}{|x-y|^{m+n}} |f(y)| \, dy,
$$

thus, the lemma follows from [6, 7].
4 Proof of Theorems

We only need to prove Theorem 1.

**Proof of Theorem 1.** For $x \in \mathbb{R}^n$, let $Q = Q(x_0, d)$ be a cube centered at $x_0$ and having side length $d$ such that $x \in Q$. It is sufficient to prove for $f \in C_0^\infty(\mathbb{R}^n)$ and some constant $C_0$, that the following inequality holds:

$$\left( \frac{1}{|Q|} \int_Q \| g_\psi^A(f)(x) - C_0 \|^r dx \right)^{1/r} \leq CM^2(\tilde{f}(\chi)),$$

Set $\tilde{Q} = 5\sqrt{n} Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!}(D^\alpha A)\tilde{Q} x^\alpha$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)Q$ for $|\alpha| = m$. We write, for $f_1 = f\chi_Q$ and $f_2 = f\chi_{\mathbb{R}^n \setminus \tilde{Q}}$,

$$F_t^A(f)(x) = \int_{\mathbb{R}^n} \psi_t(x-y)R_{m+1}(\tilde{A}; x, y) f_2(y) dy$$

$$+ \int_{\mathbb{R}^n} \psi_t(x-y)R_m(\tilde{A}; x, y) f_1(y) dy$

$$- \sum_{|\alpha| = m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \psi_t(x-y)(x-y)^\alpha D^\alpha \tilde{A}(y) f_1(y) dy,$$

then

$$\| g_\psi^A(f)(x) - g_\psi^A(f_2)(x_0) \|$$

$$= \| F_t^A(f)(x) - F_t^A(f_2)(x_0) \|$$

$$\leq \| F_t^A(f)(x) - F_t^A(f_2)(x_0) \|$$

$$\leq \| F_t \left( \frac{R_m(\tilde{A}; x, y)}{|x-y|^m} f_1 \right) (x) \|$$

$$+ \sum_{|\alpha| = m} \frac{1}{\alpha!} \| F_t \left( \frac{(x-y)^\alpha D^\alpha \tilde{A}f_1}{|x-y|^m} \right) (x) \|$$

$$\equiv I(x) + II(x) + III(x),$$

thus,

$$\left( \frac{1}{|Q|} \int_Q \| g_\psi^A(f)(x) - g_\psi^A(f_2)(x_0) \|^r dx \right)^{1/r}$$

$$\leq \left( \frac{C}{|Q|} \int_Q I(x)^r dx \right)^{1/r} + \left( \frac{C}{|Q|} \int_Q II(x)^r dx \right)^{1/r} + \left( \frac{C}{|Q|} \int_Q III(x)^r dx \right)^{1/r}$$

$$\equiv I + II + III.$$

Now, let us estimate $I$, $II$ and $III$, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, using Lemma 2, we get

$$R_m(\tilde{A}; x, y) \leq C|x-y|^m \sum_{|\alpha| = m} \| D^\alpha A \|_{BMO},$$

139
thus, by Lemma 1 and the weak type (1,1) of $g_\psi$ (see [11, 16]), we obtain

$$I \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} |Q|^{-1} \frac{\|g_\psi(f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^r/(1-r)}}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} |Q|^{-1} \|g_\psi(f_1)(f_1)\|_{W^{1,1}}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} |Q|^{-1} \int_Q |f(y)| dy$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} M(f)(\tilde{x});$$

For $II$, similar to the proof of $I$, we get

$$II \leq C \sum_{|\alpha|=m} |Q|^{-1} \frac{\|g_\psi(D^\alpha \tilde{A}f_1)\chi_Q\|_{L^r}}{\|\chi_Q\|_{L^r/(1-r)}}$$

$$\leq C \sum_{|\alpha|=m} |Q|^{-1} \|g_\psi(D^\alpha \tilde{A}f_1)\|_{W^{1,1}}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{exp} L, Q} \|f\|_{L \log L, \tilde{Q}}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO} M L \log L, f(\tilde{x})}$$

$$\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO} M^2(f)(\tilde{x})};$$

To estimate $III$, we write,

$$F_t^{\hat{A}}(f_2)(x) - F_t^{\hat{A}}(f_2)(x_0)$$

$$= \int_{\mathbb{R}^n} \left[ \frac{\psi_t(x-y)}{|x-y|^m} - \frac{\psi_t(x_0-y)}{|x_0-y|^m} \right] R_m(\tilde{A}; x, y) f_2(y) dy$$

$$+ \int_{\mathbb{R}^n} \frac{\psi_t(x_0-y) f_2(y)}{|x_0-y|^m} \left[ R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) \right] dy$$

$$- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{\mathbb{R}^n} \left( \frac{\psi_t(x-y)(x-y)^\alpha}{|x-y|^m} - \frac{\psi_t(x_0-y)(x_0-y)^\alpha}{|x_0-y|^m} \right) D^\alpha \tilde{A}(y) f(y) dy$$

$$= III_1 + III_2 + III_3.$$  

Note that $|x-y| \sim |x_0-y|$ for $x \in Q$ and $y \in \mathbb{R}^n \setminus \tilde{Q}$. By Lemma 3 and the following inequality (see [9])

$$|b_{Q_2} - b_{Q_1}| \leq C \log (|Q_2|/|Q_1|) \|b\|_{\text{BMO}}, \quad \text{for } Q_1 \subset Q_2,$$

we know that, for $x \in Q$ and $y \in 2^{k+1} \tilde{Q} \setminus 2^k \tilde{Q},$

$$|R_m(\tilde{A}; x, y)| \leq C |x-y|^m \sum_{|\alpha|=m} \left( \|D^\alpha A\|_{\text{BMO}} + |(D^\alpha A)_{Q(x, y)} - (D^\alpha A)_{Q}|ight)$$

$$\leq C k |x-y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}};$$
By the condition of $\psi$, and similar to the proof of Lemma 4, we obtain
\[
\|III_1\| \leq C \int_{\mathbb{R}^n \setminus Q} \frac{|x - x_0|}{|x_0 - y|^{m+n+1}} |R_m(\tilde{A}; x, y)||f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^{k}Q} k \frac{|x - x_0|}{|x_0 - y|^{n+1}} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=1}^{\infty} k 2^{-k} \frac{1}{|2^k Q|} \int_{2^k Q} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=1}^{\infty} k 2^{-k/2} M(f)(\tilde{x})
\]
\[
\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} M(f)(\tilde{x});
\]

For $III_2$, by the formula (see [2]):
\[
R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\beta| < m} \sum_{|\alpha| = m} \frac{1}{|\beta|} R_m(\tilde{A}; x, x_0)(x - y)^\beta
\]
and Lemma 1, we have
\[
|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)| \leq C \sum_{|\beta| < m} \sum_{|\alpha| = m} |x - x_0|^{m-|\beta|} |x - y|^{|\beta|} \|D^\alpha A\|_{\text{BMO}},
\]
similar to the estimates of $III_1$, we get
\[
\|III_2\| \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} k \frac{|x - x_0|}{|x_0 - y|^{n+1}} |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} M(f)(\tilde{x});
\]

For $III_3$, similar to the estimates of $III_1$, we get
\[
\|III_3\| \leq C \sum_{|\alpha| = m} \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} k 2^{-k} \frac{|x - x_0|}{|x_0 - y|^{n+1}} \|D^\alpha A\|_{\text{BMO}}(y) |f(y)| \, dy
\]
\[
\leq C \sum_{|\alpha| = m} \sum_{k=1}^{\infty} k 2^{-k} \|D^\alpha A\|_{\text{exp L}} 2^k Q \|f\|_{\text{Log} L^2} + \|D^\alpha A\|_{\text{BMO}} M(f)(\tilde{x})
\]
\[
\leq C \sum_{|\alpha| = m} \sum_{k=1}^{\infty} k(2^{-k} + 2^{-k/2}) \|D^\alpha A\|_{\text{BMO}} M_{\text{Log}} L(f)(\tilde{x})
\]
\[
\leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} M^2(f)(\tilde{x}).
\]

Thus,
\[
III \leq C \sum_{|\alpha| = m} \|D^\alpha A\|_{\text{BMO}} M^2(f)(\tilde{x}).
\]

This completes the proof of Theorem 1. □
From Theorem 1 and the weighted boundedness of $g_\psi$ and $M$, we may obtain the conclusion of Theorem 2.
From Theorem 1 and Lemma 3, we may obtain the conclusion of Theorem 3.

Acknowledgement. The author would like to express his deep gratitude to the referee for his valuable comments and suggestions.

References


Lanzhe Liu
College of Mathematics
Changsha University of Science and Technology
Changsha 410077
P. R. of China
lanzheliu@263.net