François Murat
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie (Paris VI)

Existence and homogenization
for the problem $-\mathrm{div}\, a(x, Du) = f$
with $a(x,\xi)$ a maximal monotone graph in ξ for every x

In memory of Jacques-Louis Lions

In this lecture I will report on recent joint work [2], [3] with Gilles Francfort and Luc Tartar on the existence of a solution for some nonlinear problem and on its homogenization, two domains in which Jacques-Louis Lions made so outstanding contributions.

We consider the problem of finding u such that

$$(1) \qquad \begin{cases} u \in W_0^{1,p}(\Omega), \\ -\,\mathrm{div}\, a(x, Du) = f \quad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

where for $x \in \Omega$ given, $\xi \to a(x,\xi)$ is not a single-valued monotone continuous function from $\mathbb{R}^N$ into $\mathbb{R}^N$, but actually a multi-valued maximal monotone graph of $\mathbb{R}^N \times \mathbb{R}^N$. In such a case, the crucial issues are on the one hand the definition of an adequate setting, in particular for what concerns measurability properties of the "graph" which will replace $a(x,\xi)$, on the other hand the choice of a proper approximation procedure of this graph allowing one to prove an existence result, and finally the statement and proof of results on maximal monotone graphs allowing one to perform the homogenization of the problem.

The only paper on these topics that we are aware of is that of V. Chiadò Piat, G. Dal Maso & A. Defranceschi [1], in which delicate measurability assumptions are made in the definition of the graph and delicate measurability selection theorems are used in the proofs. In our work we provide a definition which can be proved to be equivalent to their, but which in our opinion provides a simpler framework, and in the proofs we use only classical theorems of single-valued analysis.

Our framework is the following. We look for u and d such that

$$(2) \qquad \begin{cases} u \in W_0^{1,p}(\Omega),\ d \in L^{p'}(\Omega)^N, \\ e = Du,\ -\mathrm{div}\, d = f \quad \text{in } \mathcal{D}'(\Omega), \\ (e,d) \in \mathcal{A}. \end{cases}$$

Here Ω is a bounded open set of $\mathbb{R}^N$, p and p' are real numbers with $1 < p < +\infty$ and $p' = p/(p-1)$, f belongs to $W^{-1,p'}(\Omega)$, and the "graph" $\mathcal{A}$ is defined by

$$(3) \qquad (e,d) \in \mathcal{A} \quad \Longleftrightarrow \quad d(x) - e(x) = \phi(x, d(x) + e(x)), \ \text{ a.e. } x \in \Omega,$$

where $\phi : (x,\lambda) \in \Omega \times \mathbb{R}^N \to \phi(x,\lambda) \in \mathbb{R}^N$ is a given (single-valued) Carathéodory function which is defined on the whole of $\Omega \times \mathbb{R}^N$ and satisfies

$$(4) \qquad \begin{cases} x \to \phi(x,\lambda) \text{ is measurable on } \Omega \text{ for every } \lambda \in \mathbb{R}^N, \\ |\phi(x,\lambda_1) - \phi_(x,\lambda_2)| \le |\lambda_1 - \lambda_2|, \quad \text{a.e. } x \in \Omega,\ \forall \lambda_1, \lambda_2 \in \mathbb{R}^N. \end{cases}$$

It is easy to prove that $\mathcal{A}$ is monotone, i.e. satisfies $(d_1(x)-d_2(x))(e_1(x)-e_2(x)) \geq 0$ for every $(e_1, d_1) \in \mathcal{A}$ and $(e_2, d_2) \in \mathcal{A}$ and for almost every $x \in \Omega$ (this results from the fact that ϕ is 1-Lipschitz continuous in λ), and that $\mathcal{A}$ is maximal monotone in λ for each fixed x (this results from the fact that $\phi(x, \lambda)$ is defined for every $\lambda \in \mathbb{R}^N$).

We further assume that $\mathcal{A}$ satisfies a coerciveness and a growth condition and that $(0,0) \in \mathcal{A}$, namely that for every $(e, d) \in \mathcal{A}$ and for almost every $x \in \Omega$

$$
\begin{cases} d(x)e(x) \geq \alpha|e(x)|^p - |a(x)|, \\ d(x)e(x) \geq \beta|d(x))|^{p'} - |b(x)|, \\ (0,0) \in \mathcal{A}, \end{cases} \tag{5}
$$

for some $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$ (in the single-valued case, the second assertion of (5) is another way to state the classical growth condition $|a(x,\xi)| \leq \gamma|\xi|^{p-1} + |h(x)|$ for some $\gamma > 0$ and some $h \in L^{p'}(\Omega)$).

As far as existence is concerned, we prove in [2] the existence of a function u and of a vector field d which satisfy (2).

For what concerns the homogenization of this problem, we will prove in a forthcoming paper ([3]) the following compactness result with respect to the $H-$convergence concerning this class of graphs. From every sequence ε of graphs $\mathcal{A}_\varepsilon$ which satisfy (5) uniformly (i.e. for the same $\alpha > 0$, $\beta > 0$, $a \in L^1(\Omega)$ and $b \in L^1(\Omega)$), and which are defined by (3) through (single-valued) Carathéodory functions ϕ_ε which are defined on the whole of $\Omega \times \mathbb{R}^N$ and satisfy (4), one can extract a subsequence, still denoted by ε, and there exists a graph $\mathcal{A}_0$ in the same class, such that for every $f \in W^{-1,p'}(\Omega)$, any accumulation point (u, d) (in the weak topology of $W_0^{1,p}(\Omega) \times L^{p'}(\Omega)^N$) of the solutions $(u_\varepsilon, d_\varepsilon)$ to

$$
\begin{cases} u_\varepsilon \in W_0^{1,p}(\Omega),\ d_\varepsilon \in L^{p'}(\Omega)^N, \\ e_\varepsilon = Du_\varepsilon,\ -\operatorname{div} d_\varepsilon = f \quad \text{in } \mathcal{D}'(\Omega), \\ (e_\varepsilon, d_\varepsilon) \in \mathcal{A}_\varepsilon, \end{cases} \tag{6_ε}
$$

(observe that u_ε is bounded in $W_0^{1,p}(\Omega)$ and that d_ε is bounded in $L^{p'}(\Omega)^N$) is a solution (u_0, d_0) to (6_0). This provides a new (and in our opinion simpler) proof of the analogous result proved long ago by V. Chiadò Piat, G. Dal Maso & A. Defranceschi in [1].

References

[1] Valeria Chiadò Piat, Gianni Dal Maso & Anneliese Defranceschi,
G-convergence of monotone operators,
Ann. Inst. H. Poincaré Anal. Non linéaire, 7 (1990), 123–160.

[2] Gilles Francfort, François Murat & Luc Tartar,
Monotone operators in divergence form with x-dependent multivalued graphs,
Boll. Un. Mat. Ital., (2003), to appear.

[3] Gilles Francfort, François Murat & Luc Tartar,
Homogenization of monotone operators in divergence form with x-dependent multivalued graphs,
In preparation.