Reflexivity of spaces of weakly summable sequences

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Abstract. We deal with the space of $\Lambda$-summable sequences from a locally convex space $E$, where $\Lambda$ is a metrizable perfect sequence space. We give a characterization of the reflexivity of $\Lambda(E)$ in terms of that of $\Lambda$ and $E$ and the AK property. In particular, we prove that if $\Lambda$ is an echelon sequence space and $E$ is a Fréchet space then $\Lambda(E)$ is reflexive if and only if $\Lambda$ and $E$ are reflexive.

Reflexividad de espacios de sucesiones débilmente sumables

Resumen. Consideramos el espacio de las sucesiones $\Lambda$-sumables en un espacio localmente convexo $E$, donde $\Lambda$ es un espacio de sucesiones perfecto y metrizable. Damos una caracterización de la reflexividad de $\Lambda(E)$ en términos de la de $\Lambda$ y $E$ y de la propiedad AK. En particular, demostramos que si $\Lambda$ es un espacio escalonado y $E$ es un espacio de Fréchet entonces $\Lambda(E)$ es reflexivo si y solo si $\Lambda$ y $E$ son reflexivos.

1 Introduction

The spaces $\ell_p[E]$ and $\ell_p\{E\}$ respectively of weakly $\ell_p$-summable and absolutely $\ell_p$-summable sequences in a locally convex space $E$ were first introduced by A. Pietsch [11] in connection with the nuclearity of $E$. This allowed him also to introduce and study the absolutely $p$-summing operators. Later, in the case when $E$ is a normed space, J. S. Cohen [2] introduced the space $\ell_p(E)$ of strongly $p$-summable sequences. He used this space together with the spaces $\ell_p[E]$ and $\ell_p\{E\}$ to define the strongly and the nuclear $p$-summing operators. The definition of $\ell_p(E)$ was generalized to an arbitrary locally convex space $E$ by H. Apiola [1] in order to get new conditions for the nuclearity of $E$. H. Apiola studied the duality relations between the three spaces, namely $\ell_p[E], \ell_p\{E\}$ and $\ell_p(E)$. In [11], A. Pietsch introduced and studied also the space $\Lambda(E)$ of $\Lambda$-summable sequences in $E$, $\Lambda$ being a perfect sequence space in the sense of Köthe endowed with its normal topology. M. Florencio and P. J. Paúl [4] considered the general case where $\Lambda$ is no longer equipped with the normal topology, but with a general polar one. They obtained results on $\Lambda(E)$ such as the characterization of the AK property and then the relationship with the completion $\Lambda\hat{\otimes}_e E$ of the injective tensor product $\Lambda \otimes_e E$. In [9], the authors gave a definition of strongly $\Lambda$-summable sequences. They then reconsidered the space $\Lambda(E)$ and obtained some of its properties. They mainly described the continuous dual space of $\Lambda(E)$ in terms of strongly $\Lambda^\ast$-summable sequences in $E'$, $\Lambda^\ast$ being the $\alpha$-dual of $\Lambda$ and $E'$ the dual of $E$. In this note, we are concerned with the reflexivity of the locally convex space $\Lambda(E)$. After a section giving preliminary results and definitions, we exhibit, in section 3, a fundamental family of bounded sets in $\Lambda(E)$. This allows us to characterize its strong dual space $\Lambda(E)'_3$. In section 4, we endow the space $\Lambda(E)$ of all strongly $\Lambda$-summable sequences in $E$ with a natural topology in the spirit of [1] for
We then describe the continuous dual of \( \Lambda(E) \) in terms of weakly \( \Lambda^* \)-summable sequences of \( E' \). The section 5 is devoted to the reflexivity of \( \Lambda(E) \). We show that if \( \Lambda \) and \( E \) are Fréchet spaces, then \( \Lambda(E)' = \Lambda^*(E_{\beta}) \), where \( \Lambda(E)' \) is the subspace of \( \Lambda(E) \) consisting of the sequences which are the limit of their finite sections. The equality above turns out to be topological if \( E \) happens to be semi-reflexive. We then get that, for Fréchet spaces \( \Lambda \) and \( E, \Lambda(E) \) is reflexive if and only if \( E \) and \( \Lambda \) are reflexive and the spaces \( \Lambda(E) \) and \( \Lambda^*(E_{\beta}) \) are AK. As a consequence, whenever \( \Lambda \) is an echelon space, \( \Lambda(E) \) is reflexive if and only if \( E \) and \( \Lambda \) are. Using a result of [4], this gives that, in this case, \( \Lambda \circ E \) is reflexive if and only if \( \Lambda \) and \( E \) are.

## 2 Preliminaries

Throughout this paper, \( \Lambda \) will be a perfect sequence space and \( E \) a sequentially complete Hausdorff locally convex space. The Köthe dual space of \( \Lambda \) will be denoted by \( \Lambda^* \) while \( E' \) will stand for the topological dual of \( E \). The collection of all absolutely convex, \( \sigma(E',E) \)-closed and equicontinuous subsets of \( E' \) will be denoted by \( \mathcal{M} \), while \( \mathcal{S} \) will denotes a collection of closed, absolutely convex, normal and \( \sigma(\Lambda^*,\Lambda) \)-bounded subsets of \( \Lambda^* \) such that \( \Lambda^* \) is the union of the members of \( \mathcal{S} \) and the latter is stable by homothety. We will then consider on \( \Lambda \) the polar topology \( \tau_S \) associated with the collection \( \mathcal{S} \). This topology is generated by the seminorms

\[
P_S(\alpha) := \sup \left\{ \sum_n |\alpha_n\beta_n|, \beta = (\beta_n) \in S \right\}, \quad S \in \mathcal{S}.
\]

For an absolutely convex bounded subset \( A \) of a Hausdorff topological vector space \( F \), let us denote by \( F_A \) the subspace of \( F \) generated by \( A \). When no topology is specified on \( F_A \), it will be endowed with the gauge \( ||\cdot||_A \) of \( A \) as a norm. We will then consider without any further mention the spaces \( E_B, E_M, E_R \) and \( \Lambda_S \), where \( B \) is a bounded subset of \( E, M \in \mathcal{M}, S \in \mathcal{S} \) and \( R \) is a bounded absolutely convex subset of \( \Lambda \). For every \( M \in \mathcal{M} \), consider on \( E \) the seminorm \( P_M \) defined by

\[
P_M(x) = \sup\{|a(x)|, a \in M\}
\]

and by \( E^{(M^\perp)} \) the quotient space of \( E \) by the annihilator \( M^\perp \) of \( P_M \). It is well known (see e.g. [7, Prop. 8.6.9]) that the topological dual space \( (E^{(M^\perp)})' \), when \( E^{(M^\perp)} \) is equipped with the associated quotient norm with \( P_M \), is isometrically isomorphic to the Banach spaces \( E_M' \).

A sequence \( (x_n)_n \subseteq E \) is said to be \( \Lambda \)-summable if the series \( \sum_n \alpha_n x_n \) converges in \( E \) for all \( (\alpha_n)_n \) in \( \Lambda^* \). It is weakly \( \Lambda \)-summable if \( (a(x_n))_n \in \Lambda \), for all \( a \in E' \). The space of all \( \Lambda \)-summable sequences from \( E \) will be denoted by \( \Lambda(E) \), while that of the weakly \( \Lambda \)-summable ones will be designated by \( \Lambda^*[E] \).

Similarly, \( \Lambda^*_S[E_M'] \) will stand for the weakly \( \Lambda_S \)-summable sequences from \( E_M' \), \( S \subseteq \mathcal{S} \) and \( M \in \mathcal{M} \). Following [2] and [9], we will say that the sequence \( (x_n)_n \) is strongly \( \Lambda \)-summable if for every \( M \in \mathcal{M} \), the series \( \sum_n a_n(x_n) \) converges for all \( (a_n) \in \Lambda^*[E_M'] \). The space of all such sequences will be denoted by \( \Lambda(E) \). The three spaces are linear and, since \( \Lambda \) is perfect, the following inclusions hold: \( \Lambda(E) \subseteq \Lambda(E) \subseteq \Lambda^*[E] \).

Following [4], \( \Lambda(E) \) will be equipped with the topology \( \epsilon_{\mathcal{M},\mathcal{S}} \) generated by the family \( (\epsilon_S,\mathcal{M})_{S \in \mathcal{S}, M \in \mathcal{M}} \) of seminorms, defined by

\[
\epsilon_S,M(x) := \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a(x_n)|, a \in M, \alpha = (\alpha_n)_{n \in \mathbb{N}} \in S \right\}, \quad \forall x = (x_n)_n \in \Lambda(E).
\]

These seminorms turn out to be defined also on \( \Lambda[E] \) so that \( \Lambda(E) \) is a closed topological subspace of \( \Lambda[E] \). Both spaces will henceforth be equipped with this topology. The subspace \( \Lambda(E)_r \) (resp. \( \Lambda[E]_r \)) consisting of those sequences \( x = (x_n)_n \) belonging to \( \Lambda(E) \) (resp. to \( \Lambda[E] \)) which are limits of their finite sections.
Reflexivity of spaces of weakly summable sequences

Let \( x^{(n)} \) will come in force in the sequel. Here, if \( e_n \) is the scalar sequence whose components are all zero except the \( n^{th} \) which equals 1, then

\[
x^{(n)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \sum_{i=1}^{n} x_i e_i.
\]

Note that, if \( E \) and \((\Lambda, \tau_S)\) happen to be metrizable, then so is also \( \Lambda[E] \). Moreover, if \( E \) and \((\Lambda, \tau_S)\) are Fréchet spaces, then so are also \( \Lambda[E], \Lambda(E) \) and their closed subspaces \( \Lambda[E]\), and \( \Lambda(E)\).

We refer the reader to Section 30 of [8] and Chapter 2 of [13] for details concerning Köthe theory of sequence spaces and to [7] for the terminology and notations concerning the general theory of locally convex spaces.

All the vector spaces considered here will be spaces on the field \( \mathbb{K} \) of real or complex numbers.

3 Bounded sets of \( \Lambda(E) \)

If \( B \) and \( R \) are closed absolutely convex bounded subsets respectively of \( E \) and \( \Lambda \), set

\[
R(B) := \{(x_n)_n \in \Lambda(E) : \forall x' \in B^\circ, (x'(x_n))_n \in R\}.
\]

It is easily seen that \( R(B) \) is an absolutely convex subset of \( \Lambda(E) \) and that

\[
R(B) = \left\{(x_n)_n \in \Lambda(E) : \forall \alpha = (\alpha_n)_n \in R^\circ, \sum_n \alpha_n x_n \in B \right\}.
\]

**Proposition 1** If \( B \) and \( R \) are closed absolutely convex bounded subsets respectively of \( E \) and \( \Lambda \) with \( R \) normal, then \( R(B) \) is a bounded subset of \( \Lambda(E) \). Moreover, \( R(B) \subset \Lambda[R(B)] \).

**Proof.** It is obvious that \( R(B) \) is absolutely convex. Now, if \( x = (x_n)_n \in R(B), M \in M \) and \( S \in S \), then there are \( r, s > 0 \) so that \( M \subset rB^\circ \) and \( S \subset sR^\circ \). Hence

\[
\epsilon_{S,M}(x) = \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a(x_n)|, a \in M, \alpha = (\alpha_n)_n \in S \right\}
\]

\[
= s r \sup \left\{ \sum_{n=1}^{\infty} \frac{|\alpha_n a(x_n)|}{s} , a \in M, \alpha \in S \right\}
\]

\[
\leq s r \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a(x_n)| , a \in B^\circ, \alpha \in R^\circ \right\}
\]

\[
\leq r s.
\]

Whereby \( R(B) \) is bounded in \( \Lambda(E) \). Now, let \( x = (x_n)_n \) be an element of \( R(B) \) and \( \varphi \) a continuous linear functional on \( E_B \). Then there exists \( K > 0 \) such that, for all \( b \in B, |\varphi(b)| \leq K \). Let \( \alpha = (\alpha_n)_n \in \Lambda^* \) and \( \mu > 0 \) so that \( \alpha \in \mu R^\circ \). Since \( R^\circ \) is normal, for every \( k \in \mathbb{N} \), the \( k^{th} \) finite section \( \alpha(k) \) of \( \alpha \) belongs to \( \mu R^\circ \). Hence \( \sum_{n=1}^{k} \alpha_n x_n = \mu \sum_{n=1}^{k} \mu^{-1} \alpha_n x_n \in \mu B \) and \( \sum_{n=1}^{k} \alpha_n \varphi(x_n) \leq \mu K \). Therefore \( (x_n)_n \subset E_B \). Let \( (\epsilon_n)_n \) be a scalar sequence with \( |\alpha_n \varphi(x_n)| = \epsilon_n \alpha_n \varphi(x_n) \), for all \( n \in \mathbb{N} \). Thanks to the normality of \( R^\circ, (\epsilon_n \alpha_n)_n \in \mu R^\circ \) and therefore \( \sum_{n=1}^{k} \epsilon_n \alpha_n x_n \in \mu B \). So,

\[
\sum_{n=1}^{k} |\alpha_n \varphi(x_n)| = \sum_{n=1}^{k} \epsilon_n \alpha_n \varphi(x_n)
\]

\[
= \varphi(\sum_{n=1}^{k} \epsilon_n \alpha_n x_n) \leq \mu K.
\]

53
Thus, the series $\sum \alpha_n \phi(x_n)$ is absolutely convergent with
\[
\sum_{n=1}^{\infty} |\alpha_n \phi(x_n)| \leq \mu K,
\]
showing that $(\phi(x_n))_n \in \Lambda$. Now, if $\alpha \in R^0$ then $\sum_{n=1}^{\infty} |\alpha_n \phi(x_n)| \leq K$. That is $(\phi(x_n))_n \in K R^{\infty} = KR$. Hence $(\phi(x_n))_n \in \Lambda R$, whereby $x \in \Lambda R[E_B]$. \[\blacksquare\]

The following result characterizes the bounded subsets of $\Lambda(E)$ by means of the $R(B)$’s, when $E$ and $\Lambda$ are metrizable.

**Proposition 2** If $E$ and $\Lambda$ are metrizable. Then, for every bounded subset $B$ of $\Lambda(E)$, there exist closed absolutely convex bounded subsets $B$ and $R$ respectively of $E$ and $\Lambda$ with $R$ normal such that $B \subset R(B)$.

**Proof.** Since $E$ and $\Lambda$ are metrizable, $S$ and $M$ admit fundamental sequences respectively $(S_k)_{k \in \mathbb{N}}$ and $(M_p)_{p \in \mathbb{N}}$. As $B$ is bounded, for every $p \in \mathbb{N}$,
\[
c_{k,p} := \sup \{ \epsilon_{S_k, M_p}(x), \ x \in B \} < +\infty.
\]
Set $B_k = \bigcap_p c_{k,p} M_p^\circ$. This is a bounded subset of $E$. Hence, there are $\mu_k > 0$ such that $B := \text{absconv}(\bigcup_k \mu_k B_k)$ is still bounded. Now, consider the set
\[
R_0 := \{ (a(x_n))_n, \ a \in B^0, \ x = (x_n)_n \in B \}
\]
and $R$ the normal absolutely convex hull of $R_0$. Obviously $B \subset R(B)$. So, we only need to show that $R_0$, and then also $R$, is bounded in $\Lambda$. But for $k \in \mathbb{N}$, we have
\[
P_{S_k} ( (a(x_n))_n ) = \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a(x_n)|, \ \alpha \in S_k \right\}
= \sup \left\{ \left\| a \left( \sum_{n=1}^{\infty} \alpha_n x_n \right) \right\|, \ \alpha \in S_k \right\}.
\]
In order to conclude, it suffices to show that $A_k = \{ \sum_{n=1}^{\infty} \alpha_n x_n, \ \alpha \in S_k, \ x \in B \}$ is contained in $B_k$. But for every $p \in \mathbb{N}$, $\alpha \in S_k$ and $x \in B$,
\[
P_{M_p} \left( \sum_{n=1}^{\infty} \alpha_n x_n \right) = \sup \left\{ \left\| a \left( \sum_{n=1}^{\infty} \alpha_n x_n \right) \right\|, \ a \in M_p \right\}
\leq \sup \left\{ \sum_{n=1}^{\infty} |\alpha_n a(x_n)|, \ a \in M_p \right\}
\leq c_{k,p}.
\]
Showing that $A_k \subset B_k$. \[\blacksquare\]

A slightly modified proof shows that, whenever the spaces $\Lambda$ (resp. $E$) is a normed space, the result remains true without any further condition on $E$ (resp. on $\Lambda$).

**4 Dual space of $\Lambda \langle E \rangle$**

We are going to define on $\Lambda \langle E \rangle$ a locally convex topology which extends that introduced by H. Apiola [1] in the $\ell_p$ case. We start with the following result:
Proposition 3 Let \( S \in \mathcal{S} \) and \( M \in \mathcal{M} \) be given. Then

1. \( \epsilon_{S^*,M^*} \) is a complete norm on \( \Lambda^*_S[E'_M] \), where, for \( a = (a_n)_{n \in \mathbb{N}} \in \Lambda^*_S[E'_M] \),
   \[
   \epsilon_{S^*,M^*}(a) = \sup \left\{ \sum_{n=1}^{\infty} |a_n a_n(x)|, \ x \in M^\circ, \ a = (a_n)_{n \in \mathbb{N}} \in S^\circ \right\} .
   \]

2. \( \sigma_{S,M} \) is a seminorm on \( \Lambda \langle E \rangle \), where, for all \( x = (x_n)_{n \in \mathbb{N}} \in \Lambda \langle E \rangle \),
   \[
   \sigma_{S,M}(x) = \sup \left\{ \sum_{n=1}^{\infty} |a_n(x_n)|, \ a = (a_n)_{n \in \mathbb{N}} \in \Lambda^*_S[E'_M], \ \epsilon_{S^*,M^*}(a) \leq 1 \right\} .
   \]

Proof. 1. Follows from Proposition 1 of [9], since \( \Lambda^*_S \) and \( E'_M \) are Banach spaces.

2. We only have to prove that \( \sigma_{S,M}(x) \) is finite for every \( x \in \Lambda \langle E \rangle \). Define a linear mapping \( T_x \) from \( \Lambda^*_S[E'_M] \) into \( \ell_1 \) by \( T_x((a_n)_{n \in \mathbb{N}}) = (a_n(x_n))_{n \in \mathbb{N}} \). Then \( T_x \) is continuous by the closed graph theorem. Indeed, suppose that \( (f^i)_{i \in \mathbb{N}} \in \Lambda^*_S[E'_M] \) converges to \( f := (f_n)_{n \in \mathbb{N}} \) and \( (T_x((f^i)_{i \in \mathbb{N}})) \) converges in \( \ell_1 \) to \( (\alpha_n)_{n \in \mathbb{N}} \). By the continuity of the projections, \( (f^i)_{i \in \mathbb{N}} \) converges to \( f_n \) for every \( n \in \mathbb{N} \) and then \( (f^i_n(a_n))_{i \in \mathbb{N}} \) converges to \( f_n(a_n) \) as well. It follows that \( (f_n(a_n))_{n} = (\alpha_n)_{n} \) showing that the graph of \( T_x \) is closed and then that \( T_x \) is continuous. Hence, it is bounded on the unit ball of \( \Lambda^*_S[E'_M] \). \( \square \)

From now on, the space \( \Lambda \langle E \rangle \) will be equipped with the topology \( \sigma_{S,M} \) generated by the seminorms \( \sigma_{S,M}, S \in \mathcal{S}, M \in \mathcal{M} \). We will also consider the subspace \( \Lambda \langle E \rangle_r \) of \( \Lambda \langle E \rangle \) consisting of all the sequences which are the limit of their finite sections.

The following result gives a description of the continuous dual of the subspace \( \Lambda \langle E \rangle_r \).

Theorem 1 1. For every \( S \in \mathcal{S}, M \in \mathcal{M} \) and \( a = (a_n)_{n \in \mathbb{N}} \in \Lambda^*_S[E'_M] \), the correspondence

\[
F_a : x \mapsto \sum_{n=1}^{\infty} a_n(x_n)
\]

is a continuous linear functional on \( \Lambda \langle E \rangle \).

2. Conversely, if \( F \) is a continuous linear functional on \( \Lambda \langle E \rangle \), then there exist \( S \in \mathcal{S}, M \in \mathcal{M} \) and \( a = (a_n)_{n \in \mathbb{N}} \in \Lambda^*_S[E'_M] \) so that \( F = F_a \) on \( \Lambda \langle E \rangle_r \).

3. Consequently, the topological dual \( (\Lambda \langle E \rangle_r)' \) of \( \Lambda \langle E \rangle_r \) is isomorphic to the linear space

\[
\bigcup_{S,M} \Lambda^*_S[E'_M].
\]

Proof. 1. \( F_a \) is obviously linear and for \( a = 0 \) there is nothing to show. Assume then that \( a \neq 0 \) and take \( b = \frac{a}{\epsilon_{S^*,M^*}(a)} \). Then \( \epsilon_{S^*,M^*}(b) \leq 1 \) and therefore

\[
\left| \sum_{n=1}^{\infty} a_n(x_n) \right| \leq \sum_{n=1}^{\infty} |a_n(x_n)| \leq \epsilon_{S^*,M^*}(a) \sum_{n=1}^{\infty} |b_n(x_n)| \leq \epsilon_{S^*,M^*}(a) \sigma_{S,M}(x)
\]

55
whereby $F_\alpha$ is continuous.

2. Note first that, for every $m$, the linear mapping $\theta_m$ defined from $E$ into $\Lambda\langle E\rangle$ by $\theta_m(x) = xe_m$ is continuous. Indeed, for $S \in S$ and $M \in \mathcal{M}$, one has

$$\sigma_{S,M}(\theta_m(t)) = \sup \{|u_m(t)|, \alpha \in \Lambda^*_S[E'_M], \epsilon_{S',M'}(u) \leq 1\}.$$ 

But if $\epsilon_{S',M'}(u) \leq 1$, then

$$|\alpha_m| |u_m(c)| \leq 1, \quad \forall c \in M^0, \quad \alpha = (\alpha_n)_n \in S^0.$$ 

Hence

$$|\alpha_m| \|u_m\|_M \leq 1, \quad \forall \alpha = (\alpha_n)_n \in S^0.$$ 

Fix $\alpha \in S^0$ so that $\alpha_m \neq 0$. Then

$$\sigma_{S,M}(\theta_m(t)) \leq \sup \{|u_m(t)|, |\alpha_m| \|u_m\|_M \leq 1\} \leq \sup \left\{P_M(t) \|u_m\|_M, \|u_m\|_M \leq \frac{1}{|\alpha_m|}\right\} \leq \frac{1}{|\alpha_m|} P_M(t).$$ 

Whereby $\theta_m$ is continuous. Now, since $F$ is continuous, $a_m = F \circ \theta_m$ belongs to $E'$. Moreover, there exist some $S \in S$ and some $M \in \mathcal{M}$ such that

$$|F(x)| \leq \sigma_{S,M}(x), \quad \forall x = (x_n)_n \in \Lambda\langle E\rangle.$$ 

Choosing $\alpha_m$ as above, we get

$$|a_m(t)| \leq \frac{1}{|\alpha_m|} P_M(t), \quad t \in E.$$ 

Which means that $a_n \in E'_M$. In order to show that $a = (a_n)_n \in \Lambda^*_S[E'_M]$, let $f \in (E'_M)'$, $\alpha = (\alpha_n)_n \in \Lambda$, $n \in \mathbb{N}$ and $\delta > 0$ be given. We may and do assume that $\|f\| \leq 1$. Denote by $\tilde{E}_{(M^\circ)}$ the completion of $E_{(M^\circ)}$.

Since $(\tilde{E}_{(M^\circ)})' = (E'_{(M^\circ)})'$ is isometrically isomorphic to $E'_{M}$, due to the principle of local reflexivity [3], there exists a continuous operator

$$u_n : \mathbb{K} \cdot f \rightarrow \tilde{E}_{(M^\circ)}$$

such that $\|u_k\| \leq 1 + \delta$ and $a_k(u_n f) = f(a_k)$ for all $k \in \{1, 2, \ldots, n\}$. Since every $a_n$ is continuous and $E_{(M^\circ)}$ is dense in $\tilde{E}_{(M^\circ)}$, there exist $0 < \delta_n \leq \frac{\delta}{k(1 + pS(e_n))}$ and $x_n \in E$ such that

$$\|x_k - u_n f\| \leq \delta_n \quad \text{and} \quad |a_k(x_k - u_n f)| \leq \frac{\delta}{k(|\alpha_k| + 1)},$$

$x_k$ being $x_n + M^\perp$.

We claim that the series $\sum a_n f(a_n)$ converges absolutely. So that $(f(a_n))_n$ belongs to $\Lambda^*$. We will proceed in steps:

**Step 1:** Let $\rho > 0$ be such that $\alpha$ belongs to $\rho S^0$. We have

$$\left| \sum_{k=1}^{n} a_k f(a_k) \right| \leq 2\delta + (1 + \delta)\rho, \quad n \geq 1.$$ 

56
Reflexivity of spaces of weakly summable sequences

For
\[ \left| \sum_{k=1}^{n} \alpha_k f(a_k) \right| = \left| \sum_{k=1}^{n} a_k (u_n f_k) \right| \]
\[ \leq \left| \sum_{k=1}^{n} a_k (\alpha_k x_k - u_n f_k) \right| + \left| \sum_{k=1}^{n} a_k (\alpha_k x_k) \right| \]
\[ \leq \sum_{k=1}^{n} |\alpha_k| |a_k (\overline{x}_k - u_n f)| + \left| F\left( \sum_{k=1}^{n} a_k x_k e_k \right) \right| \]
\[ \leq \delta + \sigma_{S,M} \left( \sum_{k=1}^{n} a_k x_k e_k \right) \]
\[ = \delta + \sup \left\{ \left| \sum_{k=1}^{n} x'_k (\alpha_k x_k) \right| : \ (x'_k)_{k} \in \Lambda^*_{S}[E'_M], \ \epsilon_{S^*,M^*}((x'_n)_n) \leq 1 \right\}. \]

But, for \((x'_k)_{k} \in \Lambda^*_{S}[E'_M]\) with \(\epsilon_{S^*,M^*}((x'_n)_n) \leq 1\), we have
\[ \|\alpha_k x'_n\|_{M} = \sup \{ |\alpha_k x'_k(t)| : t \in M^0 \} \]
\[ = \rho \sup \left\{ \frac{1}{\rho} |\alpha_k x'_k(t)| : t \in M^0 \right\} \]
\[ \leq \rho \epsilon_{S^*,M^*}((x'_k)_k) \]
\[ \leq \rho. \]

Whereby,
\[ \left| \sum_{k=1}^{n} x'_k (\alpha_k x_k) \right| \leq \left| \sum_{k=1}^{n} \alpha_k x'_k (\overline{x}_k - u_n f) \right| + \left| \sum_{k=1}^{n} \alpha_k x'_n (u_n f) \right| \]
\[ \leq \sum_{k=1}^{n} |\alpha_k x'_k|_{M} |\overline{x}_k - u_n f| + \sum_{k=1}^{n} |\alpha_k x'_n| (u_n f) \]
\[ \leq \sum_{k=1}^{n} \rho \delta_k + \left\| \sum_{k=1}^{n} \alpha_k x'_k \right\|_{M} \|f\| (1 + \delta) \]
\[ \leq \delta + (1 + \delta) \epsilon_{S^*,M^*}((x'_k)_k) \]
\[ \leq \delta + (1 + \delta) \rho. \]

Hence, for all \(n \in \mathbb{N}\),
\[ \left| \sum_{k=1}^{n} \alpha_k f(a_k) \right| \leq 2\delta + (1 + \delta) \rho. \]

**Step 2:** The series \(\sum \alpha_k f(a_k)\) converges absolutely.

For, since \(\alpha \in \rho S^0\) the same holds for the sequence \(\beta := (\epsilon_k \alpha_k)_k\) with \((\epsilon_n)_n\) so chosen that
\[ |\alpha_n f(a_n)| = \epsilon_n \alpha_n f(a_n), \quad n \in \mathbb{N}. \]

Then, by step 1,
\[ \left| \sum_{k=1}^{n} \epsilon_k \alpha_k f(a_k) \right| \leq 2\delta + (1 + \delta) \rho. \]
Therefore
\[\sum_{k=1}^{n} |a_k f(a_k)| = \sum_{k=1}^{n} \epsilon_k a_k f(a_k) \leq 2\delta + (1 + \delta)\rho.\]

Since \(n\) is arbitrary, the series \(\sum a_n f(a_n)\) converges absolutely. This shows that \((f(a_n))_n\) belongs to \(\Lambda^*\).

**Step 3:** \((a_n)_n\) belongs to \(\Lambda^*[E'_M]\). Indeed, since \(\delta\) is arbitrary in the last inequality, we get
\[\sum_{n=1}^{\infty} |a_n f(a_n)| \leq \rho\]
so that \((f(a_n))_k \in \rho S^{\infty} = \rho S\), whereby \((a_n)_n \in \Lambda^*[E'_M]\). Now, if \(x = (x_n)_n \in \Lambda(E)_r\) then \(x = \sum_{m=1}^{\infty} x_m e_m\) and by the continuity of \(F\) and \(F_a\) we have
\[F(x) = \sum_{m=1}^{\infty} F(x_m e_m) = \sum_{m=1}^{\infty} a_m(x_m) = F_a(x).\]

3. By 1., the map \(a \to f_a\) from \(\cup \{\Lambda^*[E'_M], S \in S, M \in M\}\) into \((\Lambda(E)_r)'\) is well defined, linear and one to one. It is onto by 2. and the definition of \(\Lambda(E)_r\). □

According to the foregoing proof, the bilinear mapping
\[\theta : \Lambda^*[E'_M] \times \Lambda(E) \to \ell_1, \langle (a_n)_n, (x_n)_n \rangle = (a_n(x_n))_n\]
is continuous in both variables.

## 5 Reflexivity of \(\Lambda(E)\)

The following lemma will be needed in the sequel:

**Lemma 1** For all \((\gamma_n)_n \in c_0\) and \(x = (x_n)_n \in \Lambda(E)\), \((\gamma_n x_n)_n \in \Lambda(E)_r\).

**Proof.** For \(S \in S\) and \(M \in M\), \((a_n)_n \in S, a \in M\) and \(p \in N\), one has
\[\sum_{n=p+1}^{\infty} |a_n a (\gamma_n x_n)| \leq \sup_{n>p} |\gamma_n| \sum_{n=p+1}^{\infty} |a_n a (x_n)| \leq \sup_{n>p} |\gamma_n| \epsilon_{S,M}(x_n)_n\]
This shows that \(\sum_{n=p+1}^{\infty} |a_n a (\gamma_n x_n)|\) converges to 0, uniformly on \(a \in M\) and \(\alpha \in S\). That is \((\gamma_n x_n)_n\) is the limit in \(\Lambda(E)\) of its finite sections which belong to \(\Lambda(E)\). The latter being closed in \(\Lambda(E)\) by Proposition 1 of [9], then \((\gamma_n x_n)_n \in \Lambda(E)_r\). □

In the sequel, \(E\) and \(\Lambda\) will be a Fréchet spaces and \(\mathcal{R}\) and \(\mathcal{B}\) the families of all absolutely convex bounded subsets of \(\Lambda\) and \(E\) respectively. The members of \(\mathcal{R}\) are assumed to be normal.

**Theorem 2** The equality \((\Lambda(E)_r)' = \Lambda^* \langle E'_\beta \rangle\) holds algebraically and the identity
\[J : (\Lambda^* \langle E'_\beta \rangle, \sigma_{\mathcal{R},\mathcal{B}}) \to (\Lambda(E)_r, \beta(\Lambda(E)_r, \Lambda(E)_r))\]
is continuous. If, in addition, \(E\) happens to be reflexive, then \(J\) turns out to be also open.
PROOF. By Theorem 7 of [9], we have

$$\langle \Lambda(E)_r \rangle' = \bigcup_{S,M} \Lambda^*_S \langle E_M' \rangle.'$$

We will then show that

$$\bigcup_{S,M} \Lambda^*_S \langle E_M' \rangle \subset \Lambda^* \langle E_{\beta} \rangle \subset (\Lambda(E)_r)' .$$

Let $S \in S$, $M \in M$ and $(a_n)_n \in \Lambda^*_S \langle E_M' \rangle$. If $H$ is an equicontinuous subset of $(E_M')'$ and $f = (f_n)_n \in \Lambda([E_{\beta}]_H')$, then The polar $H^\circ$ of $H$ with respect to the duality $\langle (E_M')', E' \rangle$ absorbs the equicontinuous (and then strongly bounded) subset $M$. There exists $\rho > 0$ such that $M \subset \rho H^\circ$. On the other hand, for all $n \in \mathbb{N}$, let $\epsilon_n > 0$ be such that $f_n \in \epsilon_n H$. Then, for all $x' \in M$, one has

$$|f_n(x')| = \rho \epsilon_n \left| \frac{1}{\epsilon_n} f_n \left( \frac{1}{\rho} x' \right) \right| \leq \rho \epsilon_n,$$

so that each $f_n$ is continuous on $E_M'$. But for $x' \in E_M'$, the mapping

$$\delta_{x'} : (E_{\beta}')'_H \to \mathbb{K}, \quad \delta_{x'}(y) = y(x')$$

is linear and continuous. Thus, $(\delta_{x'}(f_n))_n = (f_n(x'))_n \in \Lambda \subset (\Lambda^*_S)'$. Whereby $f \in (\Lambda^*_S)^*[E_M']$. By Proposition 2 of [9], since $a \in \Lambda^*_S \langle E_M' \rangle$,

$$\sum_{n=1}^{\infty} |f_n(a_n)| < \infty.$$

Hence $(a_n)_n \in \Lambda^* \langle E_{\beta}' \rangle$. Since $S$ and $M$ were arbitrary, we obtain

$$\langle \Lambda(E)_r \rangle' = \bigcup_{S,M} \Lambda^*_S \langle E_M' \rangle \subset \Lambda^* \langle E_{\beta}' \rangle.$$ 

Next, let $a = (a_n)_n \in \Lambda^* \langle E_{\beta}' \rangle$ and $(x_n)_n \in \Lambda(E)_r$. By Proposition 2, there exists absolutely convex bounded subsets $B$ of $E$ and $R$ of $\Lambda$ with $R$ normal such that $(x_n)_n \in R(B)$. Then, by Proposition 1,

$$(x_n)_n \in \Lambda R[EB] \subset \Lambda^* \langle (E_\beta') B^\circ \rangle \subset \Lambda (E_{\beta}') B^\circ,$$

where $B^\circ$ is the polar of $B^\circ$ in $(E_{\beta})'$. Since $B^\circ$ is equicontinuous, the series $\sum_{n=1}^{\infty} |a_n(x_n)|$ is convergent by the very definition of $\Lambda^* \langle E_{\beta}' \rangle$. Now, consider the linear mapping defined from $\Lambda(E)_r$ into $\ell_1$ by $\varphi_n((x_n)_n) = (a_n(x_n))_n$. Due to the closed graph theorem, $\varphi_n$ is continuous. Then the mapping $f_a : (x_n)_n \mapsto \sum_{n=1}^{\infty} a_n(x_n)$ is continuous on $\Lambda(E)_r$ and therefore belongs to $\Lambda^* \langle E_{\beta}' \rangle$. Whence $\Lambda^* \langle E_{\beta}' \rangle \subset (\Lambda(E)_r)'$.

For the second part of the proof, let $B$ be an absolutely convex bounded subset of $\Lambda(E)_r$. By Proposition 2, there exists absolutely convex bounded subsets $B$ of $E$ and $R$ of $\Lambda$ with $R$ normal such that $B \subset R(B)$. We claim that the polar $B^\circ$ of $B$ in $\Lambda^* \langle E_{\beta}' \rangle = \Lambda(E)_r$ contains the unit ball $V_{R,H}$ of $\sigma_{R,H}$; here $H = B^\circ$ is the polar of $B^\circ$ in $(E_{\beta}')'$. Let $a = (a_n)_n \in V_{R,H}$ and $x = (x_n)_n \in R(B)$. Since $B \subset H$, we have $(x_n)_n \subset (E_{\beta}')_H$. Therefore $(x'(x_n))_n \in \Lambda$ for all $x' \in E'$. By Proposition 2 of [9], we have $(x_n)_n \in \Lambda ([E_{\beta}]_H')$. But $(x'(x_n))_n \in R$, for all $x' \in B^\circ$ and

$$\epsilon_{R^\circ,B^\circ}((x_n)_n) = \sup \left\{ \sum_{n=1}^{\infty} |a_n x'(x_n)|, \; x' \in H^\circ, \; \alpha = (a_n)_n \in R^\circ \right\} \leq 1.$$
Therefore, \( \mathcal{B} \supset V_{R,H} \). To see that \( J \) is open, let \( H \) be an absolutely convex equicontinuous subset of \((E_\beta')'\). Then \( H \) is \( \sigma((E_\beta')',E_\beta) \)-bounded. Since \( E \) is semi-reflexive, there exists an absolutely convex bounded subset \( B \) of \( E \) such that, \( H = h(B) \), where \( h : E \rightarrow (E_\beta')' \) is the canonical isomorphism.

If \( R \) is a normal bounded subset of \( \Lambda \), \( R(B) \supset V_{R,H} \). Indeed, let \((a_n)_n \in R(B)^o \), \( f = (f_n)_n \in \Lambda_R[(E_\beta')]' \), with \( \epsilon_{R,H}((f_n)_n) \leq 1 \). For all \( n \in \mathbb{N} \), there exists \( x_n \in E_B \) such that \( f_n = j(x_n) \). So that, \((x_n)_n \in \Lambda[E]\). Since \( \epsilon_{R,H}((x_n)_n) \leq 1 \), for all \( a \in B^o \), \((a(x_n))_n \in R \). If \( \alpha = (a_n)_n \in R \), \( a \in B^o \) with \((\gamma_n)_n \in c_0 \) and \( |(\gamma_n)_n|_{c_0} \leq 1 \). We have

\[
\sum_{n=1}^{\infty} |\gamma_n a_n| \leq \sum_{n=1}^{\infty} \epsilon_{R,H}((x_n)_n) \leq 1.
\]

By lemma 1, \((\gamma_n x_n)_n \) is in \( \Lambda(E)_r \) and then \( R(B) \). Thus,

\[
\sum_{n=1}^{\infty} \gamma_n f_n(a_n) \leq \sum_{n=1}^{\infty} \gamma_n a_n(x_n) \leq \sum_{n=1}^{\infty} |a_n(x_n)| \leq 1.
\]

This shows that \( \sum_{n=1}^{\infty} |f_n(a_n)| \leq 1 \), and \((a_n)_n \in V_{R,H} \).

Next we prove our main result.

**Theorem 3** If \( E \) and \( \Lambda \) are Fréchet spaces, then \( \Lambda(E) \) is reflexive if, and only if, the following three assertions hold:

(i) \( E \) and \( \Lambda \) are reflexive.

(ii) \( \Lambda(E) \) is an AK-space.

(iii) \( \Lambda^*E_\beta' \) is an AK-space.

**Proof.** Suppose that \( \Lambda(E) \) is reflexive, then \( E \) and \( \Lambda \) are reflexive as subspaces of \( \Lambda(E) \). So, (i) holds. By [8, 23.5(10)], \( \Lambda(E)_r \) is reflexive as a closed subspace of \( \Lambda(E) \), it is then weakly quasi-complete by [8, 23.5(2)]. Thus, \( \Lambda(E)_r \) is weakly sequentially complete.

Let \( x = (x_n)_n \in \Lambda(E) \). Then, the sequence \((x^{(k)})_{k \in \mathbb{N}} \), consisting of the finite sections of \( x \) is contained in \( \Lambda(E)_r \) and is weakly Cauchy in it. Indeed, consider \( a \) in \( \Lambda(E)_r \). By Theorem 7 of [9], there exists a sequence \((a_n)_n \) in \( E' \) such that the series \( \sum a_n(x_n) \) converges, \((a(x^{(k)}))_k = (\sum_{k=1}^{\infty} a_n(x_n))_k \) is then a Cauchy sequence, hence \((x^{(k)})_{k \in \mathbb{N}} \) converges weakly to a limit \( y = (y_n)_n \in \Lambda(E)_r \) and it is obvious that \( x = y \) so that (ii) holds.

Now, since \( \Lambda(E)_r \) is reflexive, the same holds for its strong dual \( \Lambda^*E_\beta' \) and the argumentation above still works to prove (iii).

Conversely, assume that (i), (ii) and (iii) are satisfied. Then, since \( \Lambda \) and \( E \) are reflexive, an application of Theorem 1 and Theorem 2 gives

\[
(\Lambda(E))'' = (\Lambda(E)_r)'', \quad \text{(by (ii))}
\]

\[
= (\Lambda^*E_\beta)' = (\Lambda^*E_\beta)'_r, \quad \text{(by (iii))}
\]

\[
= \bigcup_{\mathcal{R},B} \Lambda_R[(E_\beta)'_{B=\infty}], \quad \text{(by Theorem 1)}
\]

\[
= \bigcup_{\mathcal{R},B} \Lambda_R[E_B], \quad \text{(by (i))}
\]

\[
\subseteq \Lambda[E] = \Lambda(E).
\]
Reflexivity of spaces of weakly summable sequences

The last inclusion holds by corollary 1.4 of [5]. Hence the Fréchet space $\Lambda(E)$ is semi-reflexive and then reflexive. ■

In the sequel, $\Lambda$ will stand for an echelon space defined by a Köthe matrix $(u^k)_k$. This is an increasing sequence of strictly positive sequences and

$$\Lambda := \left\{ \alpha = (\alpha_n)_n \in K^N : P_k(\alpha) = \sum_{n=1}^{\infty} u_n^k |\alpha_n| < \infty, \forall k \in N \right\}.$$

We equip $\Lambda$ with its Fréchet locally convex topology generated by the sequence $(P_k)_{k \in N}$ of seminorms.

**Proposition 4** If $\Lambda$ is reflexive then $\Lambda^* \langle E'_\beta \rangle$ is an AK-space.

**Proof.** Let $(a_n)_n \in \Lambda^* \langle E'_\beta \rangle$, we have to prove that $(a^{(k)})_k$ defined by $a^{(k)} = (0, \ldots, 0, a_{k+1}, a_{k+2}, \ldots)$, for all $k \in N$, is a null sequence. Let $R$ be an absolutely convex normal closed and bounded subset of $\Lambda$, $H$ an equicontinuous absolutely convex subset of $\|x\|_P$ such that $f \in \Lambda^*[E'_\beta]_H$ such that $\epsilon_{R^*,H}(f) \leq 1$. By Theorem 2 and the remark following ([8, 45. 5. (8)]) there exist $\gamma = (\gamma_n)_n \in \Lambda^*$ and a pre-nuclear sequence $(x'_n)_n \subset E'$ such that for all $n \in N$, $a_n = \gamma_n x'_n$. First we prove that $(f_n(x'_n))_n \in \Lambda$. Let $\alpha = (\alpha_n)_n \in \Lambda^*$, $\epsilon > 0$, $S \in \mathcal{S}$, such that $\beta \in S$ and $p \in N$. Since $(x'_n)_n$ is pre-nuclear, there exist an equicontinuous subset $M \subset E'$, and a positive Radon measure $\mu$ on $M$ such that

$$\sup_n |x'_n(x)| \leq \int_M |a(x)| \, d\mu(a).$$

As, $|x_n(x)| \leq \|x\|_P M(x)$, for all $n \in N$, $(x'_n)_n \subset E'_M$. Now, since $M$ is equicontinuous, as we did in the proof of Theorem 2, $f_n \in (E'_M)'$. Now, by the principle of local reflexivity, there exists a continuous linear operator

$$T_p : \text{span} \{f_1, f_2, \ldots, f_p\} \rightarrow E_{(M^*)}$$

such that $\|T_p\| \leq 1 + \epsilon$ and $x'_n(T_p f_n) = f_n(x'_n)$ for all $n \in \{1, 2, \ldots, p\}$. So,

$$\sum_{n=1}^{p} |\alpha_n f_n(x'_n)| = \sum_{n=1}^{p} |\alpha_n x'_n(T_p f_n)| \leq \sum_{n=1}^{p} \int_M |\alpha_n a(T_p f_n)| \, d\mu(a) \leq \|\mu\| \sup \left\{ \sum_{n=1}^{p} |\alpha_n a T_p f_n|, \quad a \in M \right\} \leq \|\mu\| \sup \left\{ \sum_{n=1}^{p} |\beta_n a T_p f_n|, \quad a \in M, (\beta_n)_n \in S \right\} \leq \rho_1 \rho \|\mu\| \epsilon_{R^*,H}(f) \leq \rho_1 \rho \|\mu\|_1,$$

where $\rho_1$ is such that $S \subset \rho_1 R^o$. Hence, $(f_n(x'_n))_n \in \Lambda$. Without loss of generality, we (may and do) assume that $(f_n(x'_n))_n \in R$. Hence,

$$\left\{ (f_n(x'_n))_n, \ f = (f_n)_n \in \Lambda_R \left[ (E'_\beta)'_H \right], \epsilon_{R^*,H}(f) \leq 1 \right\} \subset R,$$

which is $\sigma(\Lambda, \Lambda^*)$-compact, since $\Lambda$ is reflexive. It follows from [13, 2.4.26] that

$$\lim_{k \rightarrow \infty} \sup \left\{ \sum_{n=k+1}^{\infty} |\gamma_n f_n(x'_n)|, \ f = (f_n)_n \in \Lambda_R \left[ (E'_\beta)'_H \right], \epsilon_{R^*,H}(f) \leq 1 \right\} = 0.$$

Thus, $\lim_{k \rightarrow \infty} \sigma_{R,H}(a^{(k)}) = 0$. This finishes the proof. ■
Theorem 4 Let $\Lambda$ be an echelon sequence space and $E$ a Fréchet space. Then $\Lambda(E)$ is reflexive if and only if $\Lambda$ and $E$ are reflexive.

PROOF. It derives from Theorem 3, Proposition 4 and the fact that $\Lambda(E)$ is an AK-space by [8, 44.8 (10)]. □

Corollary 1 For any echelon space $\Lambda$ and any Fréchet space $E$, the injective tensor product $\Lambda \hat{\otimes} E$ is reflexive, if and only $\Lambda$ and $E$ are reflexive.

PROOF. It follows from the Proposition 2 of [4] and the preceding theorem. □

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