Holomorphically Dependent Generalised Inverses

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Abstract. In this article we investigate when the pointwise existence of a generalised inverse for holomorphic operator-valued mappings defined on domains in a Banach space implies the existence of a holomorphic generalised inverse.

1 Introduction

Let $f$ denote a holomorphic mapping from a domain $\Omega$ in a Banach space into $\mathcal{L}(X, Y)$, the space of continuous linear mappings from the Banach space $X$ into the Banach space $Y$. Over many years different authors, e.g. [1, 2, 4, 5, 7, 12], have considered when pointwise invertibility properties, of various kinds, imply the existence of a globally smooth inverse of the same kind. For example, if $f(z)$ has a right inverse for each $z \in \Omega$ does there exist $g$, holomorphic on $\Omega$ with values in $\mathcal{L}(Y, X)$, such that $g(z)$ is a right inverse for $f(z)$ for all $z \in \Omega$? In this paper we continue our investigations of such problems. Many results are known when $\Omega$ is a domain in a finite dimensional space and our interest is focused on the problems that arise when $\Omega$ is a domain in an infinite dimensional space.

We refer to [6, 10] for background information on operators between Banach spaces, to [3, 9] for the theory of holomorphic mappings on Banach spaces and to [6, 7, 12] for classical results on holomorphic dependence of operator-valued functions over finite dimensional complex manifolds.

2 Linear Preliminaries

If $X$ and $Y$ are Banach spaces over $\mathbb{C}$, $\mathcal{L}(X, Y)$ will denote the space of all continuous linear operators from $X$ to $Y$ and $GL(X, Y)$ will denote the set of all invertible linear operators from $X$ to $Y$. If $X$ and $Y$ are subspaces of the Banach space $Z$ we use the notation $Z = X \oplus Y$ to indicate that $X$ and $Y$ are closed complemented subspaces of $Z$ and that $Z$ is the direct sum of $X$ and $Y$. We let $\mathcal{H}(\Omega, X)$ denote the set of all $X$-valued holomorphic mappings defined on an open subset $\Omega$ of a Banach space. We also use the standard notation $\mathcal{L}(X) := \mathcal{L}(X, X)$ and $GL(X) := GL(X, X)$. 

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Definition 1. Let \( T \in \mathcal{L}(X, Y) \). If \( S \in \mathcal{L}(Y, X) \) and \( TST = T \) we call \( S \) a pseudo-inverse for \( T \). If, in addition, \( STS = S \) we call \( S \) a generalised inverse for \( T \). If \( TS = 1_Y \) we call \( S \) a right inverse for \( T \). The operator \( T \) is called splitting if \( \ker(T) \) and \( \im(T) \) are complemented in \( X \) and \( Y \) respectively.

The following proposition contains some important known results about generalised inverses ([2, 12]).

Proposition 1. If \( X \) and \( Y \) are Banach spaces and \( T \in \mathcal{L}(X, Y) \) then the following are equivalent:

(a) \( T \) has a pseudo-inverse,
(b) \( T \) has a generalised inverse,
(c) \( T \) is a splitting operator.

Right inverses are generalised inverses and generalised inverses are pseudo-inverses. If \( S \) is a pseudo-inverse for \( T \) then \( STS \) is a generalised inverse for \( T \).

We require the following construction of a generalised inverse. Let \( T \in \mathcal{L}(X, Y) \) and suppose \( X = \ker(T) \oplus X_1 \) and \( Y = Y_1 \oplus \im(T) \) are direct sum decompositions. The restriction of \( T \) to \( X_1 \), \( T_{R_1} \), is a continuous bijective linear mapping from \( X_1 \) onto \( \im(T) \) and, by the open mapping theorem, a continuous inverse, \( T_{R_1}^{-1} \). We define \( S : Y \to X \) by letting \( S(y_1 + y_2) = T_{R_1}^{-1}(y_2) \) for \( y_1 \in Y_1 \) and \( y_2 \in \im(T) \). If \( x_1 \in \ker(T) \) and \( x_2 \in X_1 \) then
\[
TST(x_1 + x_2) = TST(x_2) = T(T_{R_1}^{-1}T(x_2)) = T(x_2) = T(x_1 + x_2)
\]
and \( TST = T \). Moreover, if \( y_1 \in Y_1 \) and \( y_2 \in \im(T) \), then
\[
STS(y_1 + y_2) = S(TT_{R_1}^{-1}(y_2)) = S(y_2) = S(y_1 + y_2),
\]
and \( S \) is a generalised inverse for \( T \).

Lemma 1. If \( P \) and \( Q \) are projections in \( \mathcal{L}(X) \) and \( \|P - Q\| < 1 \) then \((1_X - P + Q) \in GL(X)\) and \((1_X - P + Q)(P(X)) = Q(X)\). In particular, \( P(X) \simeq Q(X)\).

Proof. Let \( R := 1_X - P + Q \). Since \((1_X - P + Q)P = PQ\) we have
\[
R(P(X)) = (1_X - P + Q)(P(X)) \subseteq Q(X). \tag{1}
\]
Since \( \|P - Q\| < 1 \), \( R := 1_X - P + Q \in GL(X) \) and
\[
R^{-1} = (1_X - P + Q)^{-1} = \sum_{n=0}^{\infty} (P - Q)^n = \left[ \sum_{n=0}^{\infty} (P - Q)^{2n} \right] (1_X + P - Q).
\]
Interchanging \( P \) and \( Q \) in (1) we obtain \((1_X - Q + P)(Q(X)) \subseteq P(X)\) and as \((P - Q)^2 P = P(1_X - QP)\) we see that \((P - Q)^2 P(X) \subseteq P(X)\). Hence \( R^{-1}(Q(X)) \subseteq P(X) \) and \( Q(X) \subseteq R(P(X)) \). Combining this with (1) completes the proof. \( \square \)

3 Vector Bundles

In this section we recall the definition of holomorphic Banach vector bundles and generalise to Banach spaces a result of Shubin [11] (see also [12, Theorem 3.11]).

Let \( \pi : E \to \Omega \) be a surjective holomorphic map of complex Banach manifolds. We assume that the fibre above \( z \in \Omega \), \( E_z := \pi^{-1}(z) \), has been given a Banach space structure whose topology coincides with the topology induced from \( E \). A collection \((U_\alpha, \tau_\alpha)_{\alpha \in \Lambda}\) is called a trivialising cover for \( \pi \) if \((U_\alpha)_{\alpha \in \Lambda}\) is an open cover of \( \Omega \) and for each \( \alpha \in \Lambda \) there is a Banach space \( X_\alpha \) such that \( \tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha \) is a biholomorphic mapping and conditions (i), (ii) and (iii) below are satisfied.
(i) $\tau_{\alpha,z} := \tau_{\alpha}|E_z$ is a linear isomorphism$^1$, from $E_z$ onto $X_\alpha$ for each $z \in U_\alpha$.

(ii) $\pi|_{\tau^{-1}(U_\alpha)} = \pi_\alpha \circ \tau_\alpha$, where $\pi_\alpha$ is the canonical projection from $U_\alpha \times X_\alpha$ onto $U_\alpha$.

Conditions (i) and (ii) imply that $\rho_{\alpha\beta} := \tau_{\alpha} \circ \tau^{-1}_{\beta}|_{U_{\alpha}\times X_\beta}$ has the form $\rho_{\alpha\beta}(z,x) = (z, g_{\alpha\beta}(z)x)$, where $g_{\alpha\beta}(z) \in \mathcal{L}(X_\beta, X_\alpha)$ and $x \in X_\beta$ whenever $\alpha, \beta \in \Lambda$ and $z \in U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$.

(iii) If $\alpha, \beta \in \Lambda$ and $U_\alpha \cap U_\beta \neq \emptyset$ then the map $z \mapsto g_{\alpha\beta}(z)$ from $U_{\alpha\beta}$ into $\mathcal{L}(X_\beta, X_\alpha)$ is holomorphic.

Two trivialising covers are said to be equivalent if their union is also a trivialising cover.

**Definition 2.** A holomorphic vector bundle is a triple $(\mathcal{E}, \pi, \Omega)$, where $\pi : \mathcal{E} \to \Omega$ is a surjective holomorphic map of complex Banach manifolds, together with a class of equivalent trivialising covers for $\pi$.

We call $\mathcal{E}$ the **bundle space**, $\pi$ the **projection** of the bundle, $\Omega$ the **base** of the bundle, $\{\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha\}$, $(U_\alpha, \tau_\alpha, X_\alpha)$, $(U_\alpha, \tau_\alpha)$ or just $\tau_\alpha$ a **trivialization** of $\pi^{-1}(U_\alpha)$ and $g_{\alpha\beta}$ a transition map. Note that $g_{\alpha\alpha}(z) = 1_{X_\alpha}$ for all $z \in U_\alpha$, $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, on $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$, and $g_{\alpha\beta}(z)^{-1} = g_{\beta\alpha}(z)$ for all $z \in U_{\alpha\beta}$.

For convenience, we often write $\mathcal{E}$ in place of $(\mathcal{E}, \pi, \Omega)$.

If $X$ is a Banach space and $\Omega$ is a complex manifold, the triple $(\Omega \times X, \pi, \Omega)$, where $\pi$ is the canonical projection from $\Omega \times X$ onto $\Omega$, together with the covering trivialisation $(1_{\Omega \times X}, \Omega \times X \to \Omega \times X)$ is called the **trivial bundle**. If $\mathcal{E}$ is a holomorphic vector bundle and $(U, \tau, X)$ is a trivialisation of $\pi^{-1}(U)$ then $E_U := (\pi^{-1}(U), \pi|_{\pi^{-1}(U)}, U)$ is a trivial bundle with covering trivialisation $(U, \tau, X)$.

A **holomorphic section** of the holomorphic vector bundle $(\mathcal{E}, \pi, \Omega)$ is a holomorphic mapping $f : \Omega \to \mathcal{E}$ such that $\pi \circ f = 1_{\Omega}$. We let $\Gamma(\mathcal{E})$ denote the set of all holomorphic sections of $\mathcal{E}$. For any complex manifold $\Omega$ and any Banach space $X$, $\Gamma(\Omega \times X) \simeq \mathcal{H}(\Omega, X)$.

In proving the main result in this section we require the following important theorem of Lempert [8].

**Theorem 1.** Let $Z$ be a Banach space with an unconditional basis, $\Omega \subset Z$ pseudo-convex open, $\mathcal{E}$ a holomorphic Banach vector bundle, then the sheaf cohomology groups $H^q(\Omega, \mathcal{E})$ vanish for all $q \geq 1$.

Let $(U_\alpha)_{\alpha \in \Gamma}$ be an open covering of $\Omega$. A **Cousin data** for $(U_\alpha)_{\alpha \in \Gamma}$ is a collection of functions $f_{\alpha\beta} \in \mathcal{H}(U_{\alpha\beta}, \mathcal{E})$ satisfying $f_{\alpha\beta} + f_{\beta\alpha} = 0$ on $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$, and $f_{\alpha\beta} + f_{\beta\gamma} + f_{\gamma\alpha} = 0$ on $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$ whenever $U_{\alpha\beta\gamma} \neq \emptyset$. The **additive Cousin problem** consists in finding $f_\alpha \in \mathcal{H}(U_\alpha, \mathcal{E})$, for all $\alpha$, such that

$$f_\alpha|_{U_{\alpha\beta}} - f_\beta|_{U_{\alpha\beta}} = f_{\alpha\beta}$$

whenever $U_{\alpha\beta} \neq \emptyset$. Since the Cousin data form a 1-cocycle, when $q = 1$ Theorem 1 implies the following result.

**Corollary 1.** Let $Z$ be a Banach space with an unconditional basis, $\Omega$ be a pseudo-convex open subset of $Z$, and $(\mathcal{E}, \pi, \Omega)$ a holomorphic Banach vector bundle. If $(U_\alpha)_{\alpha \in \Gamma}$ is an open cover of $\Omega$ and $f_{\alpha\beta} \in \mathcal{H}(U_{\alpha\beta}, \mathcal{E})$ is a Cousin data then the corresponding Cousin problem is solvable.

**Example 1.** If $(\mathcal{E}, \pi, \Omega)$ is a holomorphic vector bundle we let $\mathcal{L}(\mathcal{E}) = \bigcup_{z \in \Omega} \mathcal{L}(E_z)$ and let $\theta(T_z) = z$ for all $T_z \in \mathcal{L}(E_z)$. Then $\theta : \mathcal{L}(\mathcal{E}) \to \Omega$ is surjective and $\theta^{-1}(\{z\}) = \mathcal{L}(E_z) \subset \mathcal{L}(\mathcal{E})$. We endow $\mathcal{L}(\mathcal{E})_z$ with the Banach space structure from $\mathcal{L}(E_z)$. Let $\{\tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha\}_{\alpha \in \Lambda}$ be a trivialising cover for $\mathcal{E}$. For $z \in U_\alpha$ and $T_z \in \mathcal{L}(E_z)$ let

$$\hat{\tau}_z : \theta^{-1}(U_\alpha) \to U_\alpha \times \mathcal{L}(X_\alpha)$$

be a bijective mapping and $\hat{\tau}_{\alpha,z} : \mathcal{L}(E)_z \to \mathcal{L}(X_\alpha)$ is a continuous linear mapping for all $z \in U_\alpha$. If

$^1$Here and elsewhere we identify, when necessary, $\{z\} \times X_\alpha$ and $X_\alpha$.
then, for \( z \in U_{\alpha \beta} \) and \( T \in L(X_\beta) \), we have
\[
\hat{\tau}_{\alpha \beta}(z, T) = (z, \rho_{\alpha \beta}(z) \circ T \circ g_{\beta \alpha}(z))
\]
where, as previously, \( \rho_{\alpha \beta} \), and the transition mappings \( g_{\alpha \beta} \) are defined by
\[
\rho_{\alpha \beta}(z, x) := \tau_{\alpha} \circ \tau_{\beta}^{-1}(z, x) =: (z, g_{\alpha \beta}(z)x)
\]
for \( z \in U_{\alpha \beta} \) and \( x \in X_\beta \). This implies that \( \hat{\tau}_{\alpha \beta} \) is biholomorphic for all \( \alpha, \beta \in \Lambda \) whenever \( U_{\alpha \beta} \neq \emptyset \). The bijective mappings \( \hat{\tau}_{\alpha \beta} \) can now be used with (2) to define a unique complex manifold structure on \( L(E) \) such that \( \hat{\tau}_{\alpha} : \theta^{-1}(U_\alpha) \to U_\alpha \times L(X_\alpha) \) is biholomorphic for all \( \alpha \) and such that \( (L(E), \theta, \Omega) \) is a holomorphic vector bundle with trivialising cover \((U_\alpha, \hat{\tau}_{\alpha})_{\alpha \in \Lambda}\). This bundle has transition maps \( \hat{\tau}_{\alpha \beta} \in H(U_{\alpha \beta}, L(L(X_\beta), L(X_\alpha))) \) where
\[
\left[ \hat{g}_{\alpha \beta}(z) \right](T) = g_{\beta \alpha}(z) \circ T \circ g_{\alpha \beta}(z)
\]
for \( z \in U_{\alpha \beta} \) and \( T \in L(X_\beta) \).

A sub-bundle of \((E, \pi, \Omega)\) is a bundle \((F, \eta, \Omega)\) where \( F \) is a subset of \( E \), \( \eta = \pi|_F \), \( F_z \) is a closed subspace of \( E_z \) for all \( z \in \Omega \) and the following condition holds:

There exists a trivialising cover \( \{ \tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha \}_{\alpha \in \Lambda} \) for \( E \), and a collection of Banach spaces \( (Y_\alpha)_{\alpha \in \Lambda}, Y_\alpha \subset X_\alpha \) such that \( \{ \tau_{\alpha |_{\eta^{-1}(U_\alpha)}} : \eta^{-1}(U_\alpha) \to U_\alpha \times Y_\alpha \}_{\alpha \in \Lambda} \) is a trivialising cover for \( F \).

Note that a sub-bundle is defined locally, that is given a bundle \((E, \pi, \Omega)\) and an open cover of \( \Omega, (U_\alpha) \), and for each \( \alpha \) a sub-bundle \( F_\alpha \) of \( E_{|U_\alpha} \), then there exists a unique sub-bundle \( F \) of \( E \) such that \( F_{|U_\alpha} = F_\alpha \).

This means that we may and do identify \( Y_\alpha \) with a subspace of \( X_\alpha \) and, moreover, that \( [g_{\alpha \beta}(z)]Y_\beta = Y_\alpha \) for the transition functions \( g_{\alpha \beta} \) where \( z \in U_{\alpha \beta} \) and \( \alpha, \beta \in \Lambda \). If each \( Y_\alpha \) is a complemented subspace of \( X_\alpha \), the sub-bundle is called a direct sub-bundle.

Sub-bundles can also be characterised by using transition functions. Suppose we are given a trivialising cover \( \{ \tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha \}_{\alpha \in \Lambda} \) for \( E \) with transition functions \( (g_{\alpha \beta})_{\alpha \beta \in \Lambda} \), and a collection of Banach spaces \( (Y_\alpha)_{\alpha \in \Lambda}, Y_\alpha \subset X_\alpha \) such that \( [g_{\alpha \beta}(z)]Y_\beta \subset Y_\alpha \) for all \( \alpha, \beta \in \Lambda \) and all \( z \in U_\alpha \cap U_\beta \). Since \( g_{\alpha \beta}(z)^{-1} = g_{\beta \alpha}(z) \) this implies
\[
[g_{\alpha \beta}(z)]Y_\beta = Y_\alpha
\]
for all \( z \in U_{\alpha \beta} \). Let \( F = \cup_{\alpha \in \Lambda} \pi^{-1}(U_\alpha \times Y_\alpha), \eta = \pi|_F \) and \( \varphi_\alpha = \tau_{\alpha |_{\eta^{-1}(U_\alpha)}} \) for all \( \alpha \in \Lambda \). Then \( \varphi_{\alpha \beta ; \eta^{-1}(z)} = F_z \to \{ z \} \times Y_\alpha \) is bijective and the Banach space \( E_z \) induces on \( F_z \) a Banach space structure. Since each \( \varphi_\alpha \) is the restriction of a bijective mapping it also is bijective onto its image and as \( \varphi_{\alpha \beta} := \varphi_\alpha \circ \varphi_{\beta}^{-1}(z, y) = (z, g_{\alpha \beta}(z)y) \) for all \( (z, y) \in U_{\alpha \beta} \times Y_\beta \) we see, by (3), that \((F, \varphi, \Omega)\) is a holomorphic vector bundle with trivialising cover \( \{ \varphi_\alpha : \eta^{-1}(U_\alpha) \to U_\alpha \times Y_\alpha \}_{\alpha \in \Lambda} \). Since \( \varphi_\alpha = \tau_{\alpha |_{\eta^{-1}(U_\alpha)}} \), \( F \) is a sub-bundle of \( E \).

**Example 2.** Let \((F, \eta, \Omega)\) be a sub-bundle of the holomorphic vector bundle \((E, \pi, \Omega)\). By definition we can find a trivialising cover for \( \pi, \{ \tau_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha \}_{\alpha \in \Lambda} \) and a collection of Banach spaces \( (Y_\alpha)_{\alpha \in \Lambda}, Y_\alpha \subset X_\alpha \) such that \( \{ \tau_{\alpha |_{\eta^{-1}(U_\alpha)}} : \eta^{-1}(U_\alpha) \to U_\alpha \times Y_\alpha \}_{\alpha \in \Lambda} \) is a trivialising cover for \( \eta \). Let \((L(E), \theta, \Omega)\) denote the holomorphic vector bundle with trivialising cover \( \{ \hat{\tau}_\alpha : \theta^{-1}(U_\alpha) \to U_\alpha \times L(X_\beta) \}_{\alpha \in \Lambda} \) constructed in Example 1.

For each \( \alpha \in \Lambda \) let
For \( \alpha, \beta \in \Lambda, z \in U_{\alpha \beta} \) and \( T \in Z_\beta \) we have
\[
[g_{\alpha \beta}(z)(T)](X_\alpha) \subset g_{\alpha \beta}(z) \circ T(g_{\beta \alpha}(z)X_\alpha)
\subset g_{\alpha \beta}(z) \circ T(X_\beta)
\subset g_{\alpha \beta}(z)(Y_\beta)
\subset Y_\alpha
\]
and
\[
[g_{\alpha \beta}(z)(T)](Y_\alpha) \subset g_{\alpha \beta}(z)(T(Y_\beta)) = \{0\}.
\]
Hence \( g_{\alpha \beta}(z)(Z_\beta) \subset Z_\alpha \) for all \( z \in U_{\alpha \beta} \). This implies, following our discussion above, that \( L(E \otimes F) := \bigcup_{\alpha \in \Lambda} g_{\alpha^{-1}} \) can be endowed with the structure of a sub-bundle of \( L(E) \).

An endomorphism of the holomorphic vector bundle \( (E, \pi, \Omega) \) is a holomorphic mapping \( f: E \to E \) such that \( f \circ \pi = \pi \), \( f_z := f|_{E_z} \) is a continuous linear mapping for all \( z \in \Omega \), and the mapping
\[
z \in U \longrightarrow \tau_z \circ f_z \circ \tau_z^{-1} \in L(X)
\]
is holomorphic for any trivialising map \( \tau: \pi^{-1}(U) \to U \times X \). We denote by \( \mathcal{M}(E) \) the set of all endomorphisms of \( E \). If \( f_z^2 = f_z \) for all \( z \in \Omega \) we call \( f \) a projection.

Using the notation of Examples 1 and 2 we see that the mapping
\[
\theta: \mathcal{M}(E) \longrightarrow \Gamma(L(E)), \quad [\theta(A)](z) := A|_{E_z}
\]
is bijective and, moreover, if \( F \) is a sub-bundle of \( E \) then
\[
A(E) \subset F \iff [\theta(A)](z)E_z \subset F_z \quad \text{for all } z \in \Omega
\]
and
\[
A(F) = \{0\} \iff [\theta(A)](z)\{0\} \subset F_z \quad \text{for all } z \in \Omega.
\]
Clearly \( A \in \mathcal{M}(E) \) is a projection if and only if \( [\theta(A)](z) \) is a (linear) projection for all \( z \in \Omega \). For the trivial bundle, \( \mathcal{M}(\Omega \times X) \simeq \mathcal{H}(\Omega, L(X)) \).

**Proposition 2.** Let \( \Omega \) be a pseudo-convex open subset of a Banach space with an unconditional basis. If \( F := (F, \eta, \Omega) \) is a sub-bundle of the holomorphic vector bundle \( (E, \pi, \Omega) \) then \( F \) is a direct sub-bundle if and only if \( \exists \) a projection \( p \in \mathcal{M}(E) \) such that \( p(E) = F \).

**Proof.** We first suppose that \( F \) is a direct sub-bundle of \( E \). Let \( \{\tau_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times X_\alpha\}_{\alpha \in \Lambda} \) denote a trivialising cover for \( E \) such that \( \{\tau_\alpha|_{\pi^{-1}(U_\alpha)}: \eta^{-1}(U_\alpha) \to U_\alpha \times X_\alpha\}_{\alpha \in \Lambda} \) is a trivialising cover for \( F \). By our hypothesis \( Y_\alpha \) is a complemented subspace of \( X_\alpha \) and we let \( P_\alpha \in \mathcal{L}(X_\alpha) \) denote a continuous projection onto \( Y_\alpha \) for each \( \alpha \in \Lambda \). For each \( \alpha \) let \( E_\alpha \) denote the holomorphic vector bundle \( \pi^{-1}(U_\alpha), \pi|_{\pi^{-1}(U_\alpha)}, U_\alpha \) with trivialising cover \( (U_\alpha, \tau_\alpha, X_\alpha) \). Then \( F_\alpha := (\eta^{-1}(U_\alpha), \eta|_{\eta^{-1}(U_\alpha)}, U_\alpha) \) with trivialising cover \( (U_\alpha, \tau_\alpha|_{\eta^{-1}(U_\alpha)}, Y_\alpha) \) is a direct sub-bundle of \( E_\alpha \). We define \( f_\alpha: E_\alpha \to E_\alpha \) as follows: if \( z \in U_\alpha \) let \( f_{\alpha,z} := f_{\alpha,z} \) where
\[
f_{\alpha,z}(\xi) = \tau_{\alpha,z}^{-1} \circ P_\alpha \circ \tau_{\alpha,z}(\xi)
\]
for all $\xi \in \mathcal{E}_z$. Then $f_{\alpha,z} \in \mathcal{L}(\mathcal{E}_z)$ is a projection with $f_{\alpha,z}(\mathcal{E}_z) = \mathcal{F}_z$ for all $z \in U_{\alpha}$. Since $\tau_{\alpha,z} \circ f_{\alpha} \circ \tau^{-1}_{\alpha} = P_{\alpha}, f_{\alpha} \in \mathcal{M}(\mathcal{E}_\alpha)$ and $f_{\alpha}(\mathcal{E}_\alpha) = \mathcal{F}_\alpha$.

If $\alpha, \beta \in \Lambda$ and $U_{\alpha\beta} \neq \emptyset$ let $f_{\alpha\beta} = f_{\alpha}|_{\mathcal{E}_\alpha} - f_{\beta}|_{\mathcal{E}_\alpha}$. Then $f_{\alpha\beta} \in \mathcal{M}(\mathcal{E}_{\alpha\beta})$ and $f_{\alpha\beta}(\mathcal{E}_{\alpha\beta}) \subset \mathcal{F}_{\alpha\beta}$.

Since $f_{\alpha}(\xi) = f_{\beta}(\xi) = \xi$ for all $z \in U_{\alpha\beta}$ and all $\xi \in \mathcal{F}_z$, $f_{\alpha\beta}(\mathcal{F}_{\alpha\beta}) = \{0\}$. By (5) we can identify $f_{\alpha\beta}$ with $g_{\alpha\beta} \in \Gamma(\mathcal{L}(\mathcal{E}_{\alpha\beta}) \Box \mathcal{F}_{\alpha\beta})$. Since $(g_{\alpha\beta})_{\alpha,\beta} \in \Lambda$ forms a 1-cocycle in the sheaf of $\mathcal{L}(\mathcal{E} \Box \mathcal{F})$-valued holomorphic germs on $\Omega$, Corollary 1 implies that there exist, for all $\alpha \in \Lambda$, $g_{\alpha} \in \Gamma(\mathcal{L}(\mathcal{E}_\alpha \Box \mathcal{F}_\alpha))$ such that

$$g_{\alpha}|_{U_{\alpha\beta}} = g_{\beta}|_{U_{\alpha\beta}} = g_{\alpha\beta}. \quad (8)$$

By (5) each $g_{\alpha}$ can be identified with $h_{\alpha} \in \mathcal{M}(\mathcal{E}_\alpha)$, satisfying $h_{\alpha}(\mathcal{E}_\alpha) \subset \mathcal{F}_\alpha$ and $h_{\alpha}(\mathcal{F}_\alpha) = 0$ and, by (8),

$$h_{\alpha}|_{\mathcal{E}_\alpha} - h_{\beta}|_{\mathcal{E}_\alpha} = f_{\alpha}|_{\mathcal{E}_\alpha} - f_{\beta}|_{\mathcal{E}_\alpha}$$

for all $\alpha, \beta \in \Lambda$ whenever $U_{\alpha\beta} \neq \emptyset$. Hence

$$(f_{\alpha} - h_{\alpha})|_{\mathcal{E}_{\alpha\beta}} = (f_{\beta} - h_{\beta})|_{\mathcal{E}_{\alpha\beta}}$$

whenever $U_{\alpha\beta} \neq \emptyset$ and the mapping

$$p(\xi) := f_{\alpha}(\xi) - h_{\alpha}(\xi)$$

for all $\xi \in \pi^{-1}(U_{\alpha})$ is well defined on $\mathcal{E}$ and belongs to $\mathcal{M}(\mathcal{E})$. Since $f_{\alpha}$ and $h_{\alpha}$ both map $\mathcal{E}_\alpha$ into $\mathcal{F}_\alpha$ for all $\alpha \in \Lambda$ it follows that $p(\mathcal{E}) \subset \mathcal{F}$ and as $f_{\alpha}(\mathcal{E}_\alpha) = \mathcal{F}_\alpha$ and $h_{\alpha}(\mathcal{F}_\alpha) = \{0\}$ this implies $p(\mathcal{E}) = \mathcal{F}$. If $z \in U_{\alpha}$ and $\xi \in \mathcal{E}_z$ then $f_{\alpha,z}(h_{\alpha,z}(\xi)) = h_{\alpha,z}(\xi), h_{\alpha,z}(f_{\alpha,z}(\xi)) = 0$, and $h_{\alpha,z}(h_{\alpha,z}(\xi)) = 0$. Hence

$$p(p(\xi)) = p(f_{\alpha,z}(\xi) - h_{\alpha,z}(\xi)) \quad = f_{\alpha,z}(\xi) - f_{\alpha,z}(h_{\alpha,z}(\xi)) + h_{\alpha,z}(h_{\alpha,z}(\xi)) \quad = f_{\alpha,z}(\xi) - h_{\alpha,z}(\xi) \quad = p(\xi).$$

This completes the proof in one direction.

Since the converse is a local result we may suppose that $\mathcal{E}$ is the trivial bundle, $\Omega \times X$, that $p \in \mathcal{H}(\Omega, \mathcal{L}(X))$ and $p(z)$ is a projection for all $z \in \Omega$. We must show that $\mathcal{F} := \{(x, z) : x = p(z)x\}$ is a direct sub-bundle of $\mathcal{E}$. Fix $w \in \Omega$, and let $X_0 := p(w)X, X_1 := (1_X - p(w))X$. For $z \in \Omega$ let

$$A(z) := p(z)p(w) + (1_X - p(z))(1_X - p(w)).$$

Since $A(w) = 1_X$ we can choose a neighbourhood of $w$, $V_w$, such that $A(z)$ is invertible on $V_w$. Then

$$A(z)(X_0) = p(z)p(w)X \subset p(z)X$$

and

$$A(z)(X_1) = (1_X - p(z))(1_X - p(w))X \quad \subset (1_X - p(z))X.$$ 

Since $A(z)$ is invertible on $V_w$ we have $A(z)(X_0 + X_1) = X$, hence $A(z)(X_0) = p(z)X$ and $A(z)(X_1) = (1_X - p(z))X$. If $B(z)$ denotes the inverse of $A(z)$ then $X_0 = B(z)(p(z)X)$ for all $z \in V_w$ and the mapping

$$V_w \times X \to V_w \times X : (z, x) \to (z, B(z)x)$$

provides the required trivialization. This completes the proof. ■

Note that we did not require pseudo-convexity or Corollary 1 for the second half of the proof.
4 Generalised Inverses

In this section we consider the following question: if \( f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y)) \) and \( f(z) \) has a generalised inverse at all points in \( \Omega \), does \( f \) have a holomorphic generalised inverse?

**Definition 3.** Let \( f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y)) \), where \( X \) and \( Y \) are Banach spaces and \( \Omega \) is an open subset of a Banach space. A mapping \( g \in \mathcal{H}(\Omega, \mathcal{L}(X, Y)) \) is called a holomorphic generalised inverse for \( f \) if \( g(z) \) is a generalised inverse for \( f(z) \) for all \( z \in \Omega \).

The following example shows that a holomorphic generalised inverse need not always exist.

**Example 3.** If \( h(z) = z1_H \), where \( H \) is a one dimensional Hilbert space, then \( h \in \mathcal{H}(\mathbb{C}, \mathcal{L}(H)) \). If \( z \neq 0 \), \( f(z) \) is invertible and we have a unique generalised inverse \( g(z) := (f(z))^{-1} = z^{-1}1_H \). Since \( \lim_{z \to 0} g(z) \) does not exist \( f \) does not have a holomorphic generalised inverse.

**Proposition 3.** Let \( f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y)) \), where \( X \) and \( Y \) are Banach spaces and \( \Omega \) is an open subset of a Banach space. Then \( f \) has a holomorphic generalised inverse if and only if there exist \( P \in \mathcal{H}(\Omega, \mathcal{L}(X)) \) and \( Q \in \mathcal{H}(\Omega, \mathcal{L}(Y)) \) such that \( P(z) \) is a continuous projection onto \( \ker(f(z)) \) and \( Q(z) \) is a continuous projection onto \( \im(f(z)) \) for all \( z \in \Omega \).

**Proof.** If \( g \) is a holomorphic generalised inverse for \( f \) then the mappings \( P \) and \( Q \), defined by letting \( P(z) := g(z) \circ f(z) \) and \( Q(z) := f(z) \circ g(z) \), are the required projection-valued holomorphic mappings.

Conversely, suppose we are given the projection-valued holomorphic mappings \( P \) and \( Q \). For convenience let \( P^*(z) = 1_X - P(z) \) and let \( I_z \) denote the natural injection from \( P^*(z)X \) into \( X \) for all \( z \in \Omega \). Let

\[
g(z) := I_z \circ (f^*(z))^{-1} \circ Q(z)
\]

where \( f^*(z) = f(z)|_{P^*(z)X} \). The linear result in the second section shows that \( g(z) \) is a generalised inverse for \( f(z) \) for all \( z \in \Omega \).

To show that \( g \) is holomorphic we fix \( w \in \Omega \) and choose \( \epsilon > 0 \) such that \( W := \{ z : ||z - w|| < \epsilon \} \subset \Omega, ||P(z) - P(w)|| < 1 \) and \( ||Q(z) - Q(w)|| < 1 \) for all \( z \in W \). Let \( U(z) = 1_X + P(z) - P(w) = 1_X - P^*(z) + P^*(w) \) and \( V(z) = 1_Y - Q(z) + Q(w) = 1_Y + Q^*(z) - Q^*(w) \) for all \( z \in W \). By Lemma 1, \( U \in \mathcal{H}(W, GL(X)), V \in \mathcal{H}(W, GL(Y)) \), \( U(z)(P^*(z)X) = P^*(w)X \) and \( V(z)(Q(z)Y) = Q(w)Y \) for all \( z \in W \). We have

\[
g(z) := (I_w \circ U(z)^{-1}) \circ (U(z) \circ (f^*(z))^{-1} \circ V(z) \circ Q(z))
\]

\[
= (I_w \circ U(z)^{-1}) \circ (V(z) \circ f(z) \circ U(z)^{-1}) \circ (V(z) \circ Q(z)).
\]

Since \( V(z) \circ Q(z) = Q(w) \circ Q(z) \) for all \( z \in W \) the mapping \( z \mapsto V(z) \circ Q(z) \) lies in \( \mathcal{H}(W, \mathcal{L}(Q(w)Y, Q(w)Y)) \). By Lemma 1, the mapping \( z \in W \mapsto I_w \circ U(z)^{-1} \) belongs to \( \mathcal{H}(W, \mathcal{L}(P^*(w)X, X)) \). It remains to show that the mapping

\[
z \mapsto k(z) := (V(z) \circ f(z) \circ U(z)^{-1})^{-1}
\]

lies in \( \mathcal{H}(W, \mathcal{L}(Q(w)Y, P^*(w)X)) \). By construction the mapping

\[
z \mapsto k^*(z) := V(z) \circ f(z) \circ U(z)^{-1}
\]

lies in \( \mathcal{H}(\Omega, GL(P^*(w)X, Q(w)Y)) \) and, as \( k(z) = (k^*(z))^{-1} \), this proves that \( k \) is holomorphic. This completes the proof.

We now present the main result in this article. Note that for \( z \in \Omega \), \( \ker(f(z)) \) is the kernel of a linear operator while \( \ker(f) \) is a holomorphic vector bundle.

**Theorem 2.** Let \( \Omega \) be a pseudo-convex open subset of a Banach space with an unconditional basis and let \( X \) and \( Y \) be Banach spaces. If \( f \in \mathcal{H}(\Omega, \mathcal{L}(X, Y)) \) has a generalised inverse for each \( z \in \Omega \), then the following conditions are equivalent:

(1) \( f \) has a holomorphic generalised inverse on \( \Omega \).

(2) There exist holomorphic projections \( P \in \mathcal{H}(\Omega, \mathcal{L}(X)) \cong \mathcal{M}(\Omega \times X) \) onto \( \ker(f) := \{ (z, x) : z \in \Omega, x \in X, f(z)x = 0 \} \) and \( Q \in \mathcal{H}(\Omega, \mathcal{L}(Y)) \cong \mathcal{M}(\Omega \times Y) \) onto \( \text{im}(f) := \{ (z, y) : z \in \Omega, y \in Y, y = f(z)x \text{ for some } x \in X \} \).

(3) \( \ker(f) \) and \( \text{im}(f) \) are direct sub-bundles of the trivial bundles \( \Omega \times X \) and \( \Omega \times Y \) respectively.

(4) For every \( w \in \Omega \) there exists a neighbourhood \( V_w \) of \( w \) and closed subspaces \( X_w \subset X \) and \( Y_w \subset Y \) such that for all \( z \in V_w \), \( \ker(f(z)) \oplus X_w = X \) and \( \text{im}(f(z)) \oplus Y_w = Y \).

\textbf{Proof.} By Proposition 3, (1) and (2) are equivalent. By Proposition 2, (2) and (3) are equivalent. By the definition of sub-bundle, (3) implies (4), and it remains to show that (4) implies (3).

Since the result is local we fix \( w \in \Omega \) and show that (3) holds on a neighbourhood \( V_w \) of \( w \). If \( z \in V_w \), \( x \in X \) and \( y \in Y_w \) let \( g(z)(x + y) = f(z)x + y \). Then \( g \in \mathcal{H}(V_w, \mathcal{L}(X + Y_w, Y)) \).

\[
\ker(g(z)) = \ker(f(z)) + \{0\} \quad \text{and} \quad \text{im}(g(z)) = \text{im}(f(z)) + Y_w = Y
\]

for all \( z \in V_w \). Hence \( g \) is surjective with complemented kernel for all \( z \in V_w \). By the proof of Proposition 1 (see also Theorem 4 in [4]), \( \ker(g) = \{ (z, x, y) \in V_w \times (X + Y_w) : f(z)x = 0, y = 0 \} \) is a direct holomorphic sub-bundle of the trivial bundle \( V_w \times (X + Y_w) \). Since \( \ker(f|_{V_w}) \cong \ker(g) \subset V_w \times (X + \{0\}) \cong V_w \times X \) this implies \( \ker(f|_{V_w}) \) is a direct sub-bundle of the trivial bundle \( V_w \times X \).

By Proposition 2 there exists a holomorphic projection \( p \in \mathcal{H}(V_w, \mathcal{L}(X)) \) such that \( \ker(f(z)) = p(z)(X) \) for all \( z \in V_w \). By Lemma 1 and, if necessary, by restricting ourselves to a smaller neighbourhood of \( w \) we have \( p(z)(X) = p(w)(X) =: Z_w \) for all \( z \in V_w \). Hence \( X = Z_w \oplus X_w \) and \( f(z)(x + y) = f(z)(y) \) for all \( z \in V_w \), all \( x \in Z_w = \ker(f(z)) \), and all \( y \in X_w \). If \( h(z) := f(z)|_{X_w} \) then \( h \in \mathcal{H}(V_w, \mathcal{L}(X_w, Y)) \).

\( h(z) \) is injective and \( \text{im}(f(z)) = \text{im}(h(z)) \) is a complemented subspace of \( Y \) for all \( z \in V_w \). By adapting the proof of Proposition 1 in [4] we see that \( \text{im}(h) = \text{im}(f|_{V_w}) \) is a complemented sub-bundle of the trivial bundle \( V_w \times Y \). Hence (4) implies (3) and this completes the proof. \( \blacksquare \)

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\textbf{References}


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