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Land valuation using a real option approach

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Abstract. This paper uses real option theory to asses the value of agricultural land that can be seeded with crops. We consider one- and two-factor models for the evolution of crop prices through time and derive a partial differential equation (PDE) for the land value. We model the potential selling decision of the land owner as a put option and incorporate it as a boundary condition in the PDE for the land price. We solve this equation numerically and show that theoretical prices are close to market land prices and that the value of the put option accounts for, at least, 25% of the total land value.

Valoración de un terreno agrícola mediante opciones reales

Resumen. En este trabajo utilizamos la teoría de opciones reales para valorar un terreno agrícola donde pueden cultivarse diferentes productos. Consideramos modelos unifactoriales y bifactoriales para la evolución del precio de estos productos y obtenemos una ecuación diferencial en derivadas parciales (EDP) para valorar el terreno. Modelizamos la posible decisión de venta del dueño del terreno como una opción put y la incorporamos en la EDP del valor de la tierra. Resolvemos numéricamente dicha ecuación y mostramos que los precios obtenidos son similares a los precios de mercado del terreno y que el valor de la opción de venta representa, al menos, un 25 % del valor total del terreno.

1 Introduction

This paper studies the valuation of agricultural land. The land price should be closely tied to the present value of future cash flows from exploiting it. The valuation should also reflect the flexibility associated with owning the land. This flexibility can be properly assessed using real options.

Real options were introduced in Finance by Brennan and Schwartz [3, (1985)] to evaluate the decision of extracting minerals. This is a classical example of capital budgeting, where the Net Present Value rule is typically used. The rule says that if the present value of the future cash flows from a project is greater than the required investment, the project should be taken, since it will increase shareholder's wealth. The implicit assumption here is that the investment decision must be made now or never. However, many projects create future opportunities, which may be a significant source of value. These opportunities can be modelled as real options. With this view, an investment decision can be considered as a call option. The value of the project is just the value of the option to invest. The exercise price is the cost of the investment (that is, the amount invested), and the gross option return is the discounted expected value of the investment returns. This option is exercised when the gross return is high enough (sufficiently higher than the exercise price, to compensate for loss of the flexibility to delay). The discounted cash flow method fails to evaluate this option correctly. So, it is not surprising that Brennan and Schwartz [3, (1985)] find that the Net Present

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Value method would lead to a non-optimal extraction of minerals, because it does not capture accurately the value from waiting to invest. This value is due to uncertainty of mineral prices and represents the opportunity cost of investing now and foregoing the option to delay investment until more information arrives.

Other authors who value the option to invest using real options are Dixit and Pindyck [9, (1994)], Bhappu and Guzman [1, (1995)], and Schwartz [25, (1997)], who determine when to open a mine using alternative stochastic processes of commodity prices.

The real options framework has also been applied to study real estate development decisions. The conventional approach to pricing a land is, again, to discount future cash flows at an appropriate discount rate, but, as Titman [26, (1985)] claims, this procedure can hardly explain why many investors keep valuable land vacant. The reason is that uncertainty raises land prices above the discounted stream of rents in its current use. Titman [26, (1985)] considers a vacant developable lot as an option to purchase a building at an exercise price equal to the construction cost, and he uses the Black-Scholes-Merton formula to value the option. He demonstrates that this option is an important determinant of the value of the land. A related work is Yamazaki [27, (2001)], that studies land prices in central Tokyo. Using a basic real option pricing model, he finds that uncertainty delays investment and leads to higher asset value.

Another strand of research in agricultural land valuation has analyzed the case where the land can be developed for alternative uses. See, for instance, McDonald and Siegel [17, (1986)], Plantinga and Miller [22, (2001)],[1] and Capozza and Li [5, (2002)]. On the empirical side, Quigg [23, (1993)] has examined the predictions of real option models using $3{,}000$ urban land transactions in Seattle. He finds that land market prices incorporate a mean premium of 6 per cent of the land value for the development option to wait. In a more recent work, Cunninghan [8, (2006)] uses a data set of parcel characteristics and property transactions also in Seattle to test whether greater price uncertainty delays the timing of development and raises land prices, and he finds support for these hypotheses.

Obviously, there can be more options embedded in land prices, such as the option to choose the use of the land. Geltner et al. [11, (1996)] model this situation as a perpetual option on the best of two assets and find that land use choice can add over 40% to land value for reasonable parameter values. More recently, Du and Hennessy [10, (2008)] investigate the value of switching crops (corn or soybean). He uses Monte Carlo simulations and finds that the average cash rent valuation for the real option approach is 11% higher than that for the conventional Net Present Value method. Another embedded option is the possibility of rotating a farmland. Some examples are Plantinga [21, (1998)], who studies the optimal timber rotation and Insley and Rollins [14, (2005)], who develop a two-factor real options model of the timber harvesting decision over infinite rotations. Finally, some researchers analyze the case where the land can be converted to a different use, for instance from agricultural to urban use. See Capozza and Helsley [4, (1990)] and Capozza and Sick [6, (1994)] for more details.

In this paper, we consider a land owner who has already made a sunk cost investment in the land that entitles him to seed crops and to receive future cash flows from them. The owner has the option to sell the land at a price in the future. This price should be higher that the present value of future cash flows. Exercising this put option means that the owner is willing to sell his land and hence close the door to all future opportunities that might be provided by the land being maintained. Clearly, the value of the land should reflect both the present value of the net revenue generated by the crops and the value of the option to abandon the project and sell the land. Our work follows that of Isgin and Forster [15, (2005)], who study this problem from an empirical point of view. They use real options pricing to measure the delays in selling farmland. Using data from Ohio, they find that, in their sample, it is usually optimal to delay the sale of the land.

For simplicity, in this paper we assume the the land can only be seeded with crops, and that there is no rotation. Although there can be many options embedded in owning the land[2], we assume that it is not optimal to quit production, and that the land cannot be transformed into a developed urban or agricultural

[1] They assume that land can be dedicated to two uses: agricultural or development, and they value the land as the discounted agricultural rents up until conversion time plus the discounted development cash flows (net of conversion costs).

[2] Option to (wait to) invest, option to stop crop production, option to switch crops, etc.

piece, because of either physical or legal constraints. Although limited, this situation is common in many places (such as Argentina). Since we value the land using numerical techniques, these complexities could easily be incorporated in the analysis. We leave a full analysis of the pricing problem as an avenue of research for later work.

The remainder of the paper is organized as follows. Section 2 proposes different processes to model crop prices and provides a partial differential equation for the land price. Section 3 describes the empirical methodology and the numerical technique used to value the land. Section 4 presents the results of the estimation and the valuation of the land. Finally, Section 5 summarizes and concludes the paper.

2 Alternative stochastic models to value the land

As mentioned before, the land price should be related to the present value of the cash flows that can be obtained from the crop plantation. These cash flows will depend on the market prices of the crops that have been seeded. Typically, some sort of mean-reversion is assumed here. Another important feature when modelling corn prices is to consider the cost of carry (convenience yield).

The classical process for the early work on real options is the geometric Brownian motion (GBM). However, when the value of these options is related to the prices of commodities, this model is not suitable, since it is well known that mean reverting processes provide a better description of the price path for many commodities (see, for example, Gibson and Schwartz [12, (1990)], Cortazar and Schwartz [7, (1994)], and Schwartz [25, (1997)]).

As noted by Schwartz [25, (1997)] in an equilibrium setting, we would expect that, when prices are relatively high, supply will increase as the higher cost producers of the commodity will enter into the market putting downward pressure on prices. Conversely, when prices are relatively low, the higher cost producers will exit the market putting upward pressure on prices. In his paper, Schwartz [25, (1997)] proposes a framework to test whether commodities prices are mean reverting. In particular, he tests whether copper, oil, and gold prices follow mean-reverting processes using different models of mean reversion. He finds that, whenever the mean-reverting process is not taken into account, the Net Present Value technique for capital budgeting makes investment decision too early compared with the optimal strategy. Hence, it seems important to detect the presence of mean reversion in commodity prices and, if so, to take it into account when valuing an asset that depends on these prices.

In this paper we follow Schwartz [25, (1997)] and use different specifications to model the evolution of crop prices. In particular, we use both one- and two-factor models, where the convenience yield is assumed to be stochastic.

This section presents the processes for crop prices and convenience yields and the corresponding partial differential equations (PDE) to price the land. As explained later, these PDEs are solved numerically due to the early exercise feature of the selling option for the land owner.

We assume that the option to sell the land can be exercised once every year. Since we use a discretization scheme with time steps of one day, the land owner will have the opportunity to exercise the option once every 260 days. The decision to sell the land will depend, among other things, on current crop prices.

We propose as a first trial a risk-neutral Geometric Brownian Motion (GBM) for crop prices, that is, a process without mean reversion and with a constant convenience yield:

$$\mathrm{d}X = (r - \delta)X\,\mathrm{d}t + \sigma_1 X\,\mathrm{d}W$$

where $X \equiv X(t)$ is the price, at time t, of one unit of crop,[3] r is the instantaneous risk-free interest rate, δ is the convenience yield, σ_1 is the volatility of the crop return and $\mathrm{d}W \equiv \mathrm{d}W(t)$ is a standard Brownian process under the risk-neutral probability measure.

[3] To shorten the notation, we have chosen to skip the argument indicating time.

There are several important issues related to this model:

- As the convenience yield is constant, the model is not able to capture changes in the term structure of futures prices. In fact, from an empirical point of view, the convenience yield can change through time.
- It implies that the volatility of futures returns is equal to the volatility of spot returns.
- It assumes that the variance of the spot price grows linearly with time.

If $V \equiv V(t)$ represents the value of the land at time t, standard no-arbitrage conditions (see Merton [18, (1973)]) imply that the value of the land is given by the following partial differential equation:[4]

$$\frac{1}{2}\sigma_1^2 X^2 V_{XX} + (r-\delta) X V_X + V_t - rV = 0$$

To obtain the land value, this equation must be solved subject to the appropriate boundary conditions.

We now consider another one-factor model where the crop price follows an Inhomogeneous Geometric Brownian Motion, since it seems logical that the drift and the diffusion of the process be homogeneous functions of degree one (see Robel [24, (2001)]).

The process is given by

$$\mathrm{d}X = \lambda(\overline{X} - X)\,\mathrm{d}t + \sigma_1 X\,\mathrm{d}W \tag{1}$$

where λ is the speed of mean reversion and $\overline{X}$ is the long-run crop price. A full description of this process can be found in the Appendix A. It is worth noting that, with this formulation, the crop price is guaranteed to take non-negative values as it follows a lognormal distribution and that the convenience yield is no longer constant as it depends on the (stochastic) crop price. This process has been used for modelling the instantaneous interest rate in Brennan and Schwartz [2, (1980)].

Robel [24, (2001)] shows how to derive the PDE for the value of a contingent claim on an IHGBM. This PDE is given by

$$\frac{1}{2}\sigma_1^2 X^2 V_{XX} + \lambda(\overline{X} - X) V_X + V_t - rV = 0$$

Next, we use a two-factor model to model the evolution of crop prices. The convenience yield, $\delta \equiv \delta(t)$, is assumed to be stochastic and is incorporated as the second factor. As before, the first factor follows a Geometric Brownian motion and the convenience yield is assumed to follow an Ornstein-Uhlenbeck (O-U) process. Following Schwartz [25, (1997)], the expression for the joint risk-neutral stochastic process is

$$\begin{aligned}\mathrm{d}X &= (r-\delta) X\,\mathrm{d}t + \sigma_1 X\,\mathrm{d}W_1\\ \mathrm{d}\delta &= \kappa(\alpha - \delta)\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2\end{aligned}$$

where $\mathrm{d}W_1\,\mathrm{d}W_2 = \rho\,\mathrm{d}t$.

In this case, Schwartz [25, (1997)] shows that the corresponding PDE is given by

$$\frac{1}{2}\sigma_1^2 X^2 V_{XX} + \frac{1}{2}\sigma_2^2 V_{\delta\delta} + \rho\sigma_1\sigma_2 X V_{X\delta} + (r-\delta) X V_X + \kappa(\alpha-\delta) V_\delta + V_t - rV = 0$$

Finally, we consider another two-factor model with the same factors as before but, now the convenience yield follows an Inhomogeneous Geometric Brownian Motion. Its expression under the risk-neutral probability measure is as follows:

$$\begin{aligned}\mathrm{d}X &= (r-\delta) X\,\mathrm{d}t + \sigma_1 X\,\mathrm{d}W_1\\ \mathrm{d}\delta &= \kappa(\alpha - \delta)\,\mathrm{d}t + \sigma_2 \delta\,\mathrm{d}W_2\end{aligned}$$

where, as before, $\mathrm{d}W_1\,\mathrm{d}W_2 = \rho\,\mathrm{d}t$.

In a similar way to the previous two-factor model, the PDE is now given by

$$\frac{1}{2}\sigma_1^2 X^2 V_{XX} + \frac{1}{2}\sigma_2^2 \delta^2 V_{\delta\delta} + \rho\sigma_1\sigma_2 X\delta V_{X\delta} + (r-\delta) X V_X + \kappa(\alpha-\delta) V_\delta + V_t - rV = 0$$

[4] In this and the following PDEs, each subindex denotes partial derivative with respect to the corresponding variable.

3 Methodology

We next present the methodology to estimate the parameters of the aforementioned models and the way to value the land solving these PDEs.

3.1 Parameter estimation of the commodity prices

This subsection describes the procedure to estimate the parameters of the different processes for the crop price and the convenience yield. One of the main issues related to this empirical implementation is that the factors in the previous models we have presented are frequently not observable. As Schwartz [25, (1997)] indicates, “in many cases the spot price of a commodity is so uncertain that the corresponding futures contract closest to maturity is used as a proxy for the spot price”. This is the proxy that will be used in this paper. In a similar way, we compute the instantaneous convenience yield using the difference between two futures prices with different maturities, as suggested by Gibson and Schwartz [12, (1990)].

The parameters of the stochastic processes for the crop prices and convenience yields will be estimated by applying the GMM (Generalized Method of Moments) technique proposed by Hansen [13, (1982)]. The processes previously presented for these variables are particular cases of the more general stochastic process

$$\mathrm{d}y = (a + by)\,\mathrm{d}t + \sigma y^{\gamma}\,\mathrm{d}W \tag{2}$$

where $y \equiv y(t)$ denotes the crop price or the convenience yield at time $t \in [0, T]$.

In more detail, we have that the models are obtained imposing some restrictions on the parameters a, b, and γ in equation (2). The following Table indicates these restrictions (in boldface) and the equivalence between these parameters and those in the processes proposed for the variables:

Process	a	b	γ
GBM process	**0**	$r-\delta$	**1**
IHGBM process	$\lambda\overline{X}$	$-\lambda$	**1**
Two-factor processes			
Crop price	**0**	$r-\delta$	**1**
Convenience yield	$\kappa\alpha$	$-\kappa$	**0**

Dividing the time interval $[0, T]$ into n subintervals of length $\Delta t = T/n$, the (Euler) discrete-time approximation of equation (2) is given as

$$y(i\Delta t) - y((i-1)\Delta t) = (a + by((i-1)\Delta t))\,\Delta t + \varepsilon(i), \qquad i = 1, 2, \ldots, n \tag{3}$$

where the residuals $\varepsilon(i)$ verify

$$E\left[\varepsilon(i)\right] = 0, \quad E\left[(\varepsilon(i))^2\right] = \sigma^2\left[y((i-1)\Delta t)\right]^{2\gamma}\Delta t, \qquad i = 1, 2, \ldots, n$$

Let $\Omega = (a, b, \sigma, \gamma)$ be a vector containing the parameters included in the Euler discretization given by equation (3). Then, for $i = 1, 2, \ldots, n$, the moment vector $f(i\Delta t, \Omega)$ is given as

$$f(i\Delta t, \Omega) = \begin{bmatrix} \varepsilon(i) \\ \varepsilon(i)y((i-1)\Delta t) \\ (\varepsilon(i))^2 - \sigma^2\left[y((i-1)\Delta t)\right]^{2\gamma}\Delta t \\ \left((\varepsilon(i))^2 - \sigma^2\left[y((i-1)\Delta t)\right]^{2\gamma}\Delta t\right) y((i-1)\Delta t) \end{bmatrix}$$

Considering the restrictions implied by equation (2), we should get $E[f(i\Delta t, \Omega)] = 0$, $i = 1, 2, \ldots, n$. Using the discretization on the interval $[0, T]$ mentioned above, the sample counterpart, $g(n, \Omega)$, of $E[f(i\Delta t, \Omega)]$ is computed as

$$g(n, \Omega) = \frac{1}{n}\sum_{i=1}^{n} f(i\Delta t, \Omega)$$

The main idea underlying the GMM methodology is finding the parameters that minimize the distance between the population moments and the sample ones. Then, the GMM estimator of the parameter vector is given as

$$\Omega_0 = \arg\min \left[g(n,\Omega)' W(n)\ g(n,\Omega) \right]$$

being $W(n)$ an appropriate positive definite weighting matrix.

We can use different alternative weighting matrices. We have chosen the proposal of Newey and West [20, (1987)] to consider the possible existence of serial autocorrelation and heteroskedasticiy in the residuals.

After having performed the GMM estimation, we can use the estimated parameters to value the land, as explained in the next section.

3.2 Valuation of the put option

As mentioned before, the land value is given by the solution of a certain PDE. As we have a Bermuda put option embedded in the land value, this PDE can not be solved analytically and we are forced to deal with numerical techniques. One alternative could be to discretize the PDE and solve numerically the corresponding difference equation. However, we have to take into account additional features as, for instance, the existence of several crops or the correlation between each crop and the corresponding convenience yield when considering the previous two-factor models. As these features complicate the numerical solution of this PDE, we have chosen to face this valuation problem by applying simulation techniques.

The Bermuda feature of the put option means that, at each exercise time, the holder of this option must choose the highest value between a) the outcome obtained if this option is exercised at this time (immediate exercise value) and b) the pay-off provided if the option is exercised in a future exercise time. In this case, this pay-off must be estimated and is known as the continuation value of the option.

A possible alternative to estimate this continuation value is the Least-Squares Monte Carlo (LSM) technique proposed by Longstaff and Schwartz [16, (2001)]. This method is based on a least-squares regression combined with the cross-section information provided by a Monte Carlo simulation. After simulating a "large enough" number of paths for the evolution of the underlying asset,[5] for any path and at each possible exercise time, a regression is performed including the following variables:

- The explanatory variables are a set of basis functions that depend on the prices of the underlying assets.[6]
- The independent variable is the (discounted) pay-off(s) we expect to receive in the future.

The continuation value of the option is given by the expected value from these regressions. The optimal exercise decision is taken by comparing this (estimated) continuation value versus the exercise value. This process is repeated, recursively for any possible exercise moment, starting from the expiration date of the option and going backward in time until the first possible exercise time. Then, for any path, we determine the exact exercise time in which it is optimal to exercise the option and the corresponding pay-off. Discounting the pay-offs obtained for all the paths at the risk-free rate until the initial day and computing the arithmetic average, the option price is obtained.

4 Empirical Application

The crops to be considered in our empirical application are wheat, soya, and corn. To have an idea of the global market for these crops in 2006, we next provide some information. China is the most important

[5] Naturally, for the two-factor models, for each crop, we will simulate the corresponding sets of paths taking into account the historical correlation between each crop price and its convenience yield.

[6] Moreno and Navas [19, (2003)] compare the performance of different polynomial basis functions and illustrate numerically that the pricing results are, in general, very robust to the type and number of basis functions. In this paper, we use Chebychev-type polynomials of degree two as basis functions.

producer of wheat. It produces 16 percent of the total amount. India and USA produce 12 percent each, and France, Australia, and Russia 6 percent each. Soya is produced in many countries with United States producing almost 50 percent of the world soya output. Brazil has a market share of 18 percent and Argentina and China account for another 10 percent each. These are the main countries in which a bad or good weather season will have a significant impact on the soya price. United States also produces a significant amount of the world output of corn sharing a 43 percent of the production. China, Brazil and Argentina produce 18 percent, 6 percent and 2 percent, respectively. Also, USA is the most important corn exporter, with 60 percent of the total exports. Additionally, Argentina, China and Brazil export 15 percent, 5 percent and 4 percent, respectively.

We use daily data obtained from CBOT futures contracts covering the period November, 1996 to March, 2006. Prices are given in USD per ton. The time series for the different prices is long enough to capture at least one change in the economic cycle, around 2001.

Figures 1–3 show the time evolution for the soya, wheat and corn prices, convenience yields and volatility for the whole sample period.[7] As we can see, all the prices and convenience yields seem to follow a mean-reverting pattern. As shown in these graphs, economic slumps are followed by a drop in the price, since the 1997 recession led to a significant drop in the price of the three crops.

The following Table includes the main statistics for the prices and convenience yields for the sample period:

	Mean	Std. Dev.	Maximum	Minimum	Median
Prices					
Wheat	116.96	16.82	162.30	82.30	115.2
Soya	216.59	44.63	387.93	153.70	208.46
Corn	90.44	11.85	130.11	68.80	87.10
Convenience yields					
Wheat	0.39	0.43	1.26	−0.89	0.40
Soya	0.04	0.44	1.54	−2.69	0.13
Corn	0.17	0.47	2.40	−2.54	0.133

The following Table reports, for each crop, the correlation between the price and the convenience yield:

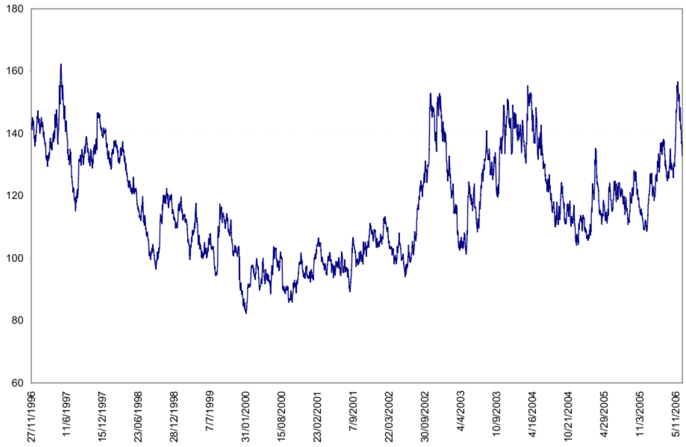
Convenience yield / Price	Wheat	Soya	Corn
Wheat	−0.375		
Soya		−0.469	
Corn			−0.395

Tables 1 and 2 report the estimated parameters (with t-values in parentheses) included in the different processes presented in the paper.

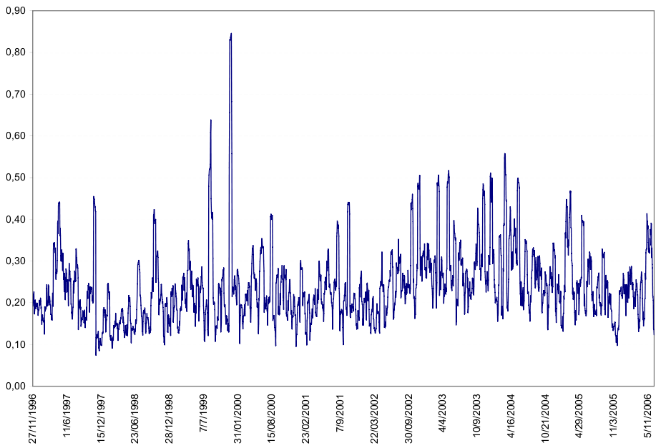
Focusing on crop prices, we can see that for the GBM process, the drift is not significantly different from zero while the diffusion coefficient is highly significant. For the IHGBM process, all the parameters are highly significant. There is a clear evidence of mean reversion for the three crops. The long-run mean values are 114.48, 149.23, and 89.48 for wheat, soya, and corn, respectively, which are quite close to the sample mean values except for soya.

With respect to convenience yields, we note that all the parameters are significant at the usual 95% confidence level except the parameter a for soya. As before, there is strong evidence of mean reversion for the three crops and the long-run mean values are close to the corresponding sample means.

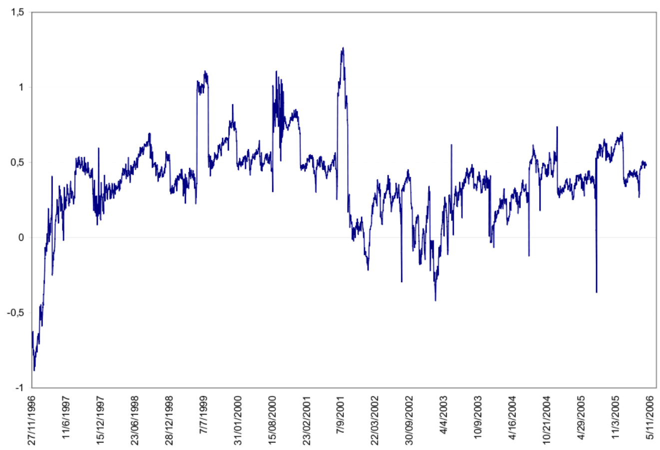
[7] For each day, the (annualized) volatility has been computed as the standard deviation of the daily rates of returns using the previous ten working days.



(a) Wheat prices.

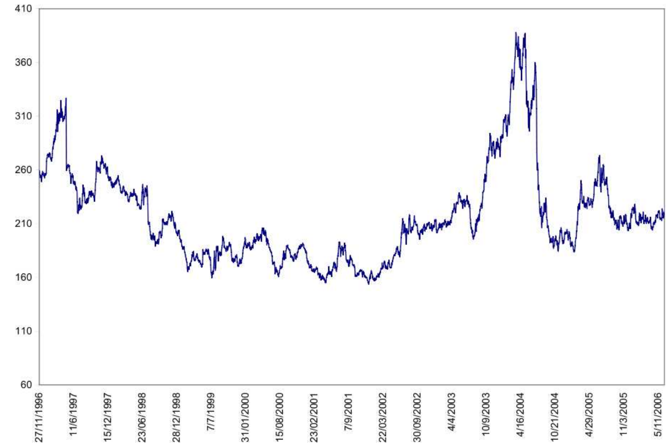


(b) Wheat volatility.

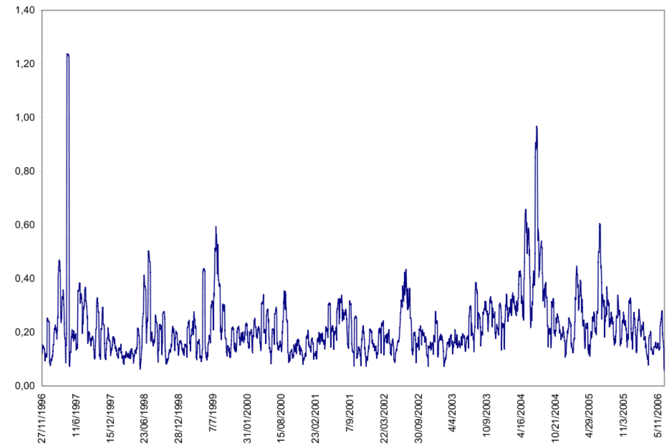


(c) Wheat convenience yield.

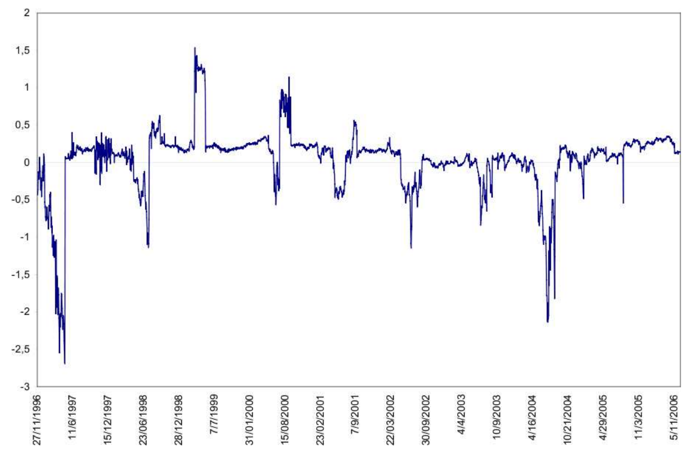
Figure 1. Wheat prices, volatility and convenience yield, from November, 1996 to March, 2006.



(a) Soya prices.

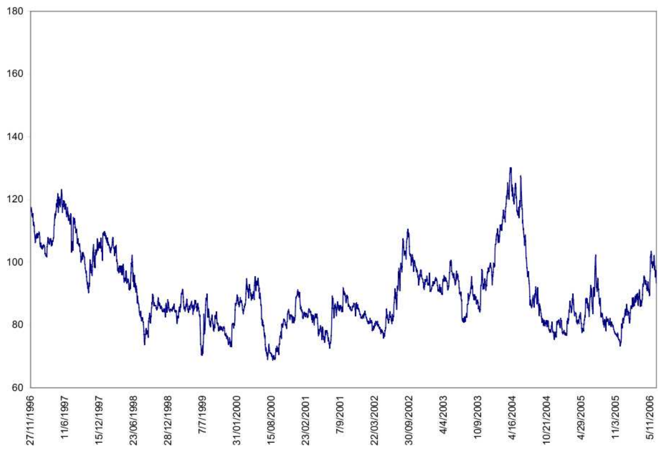


(b) Soya volatility.

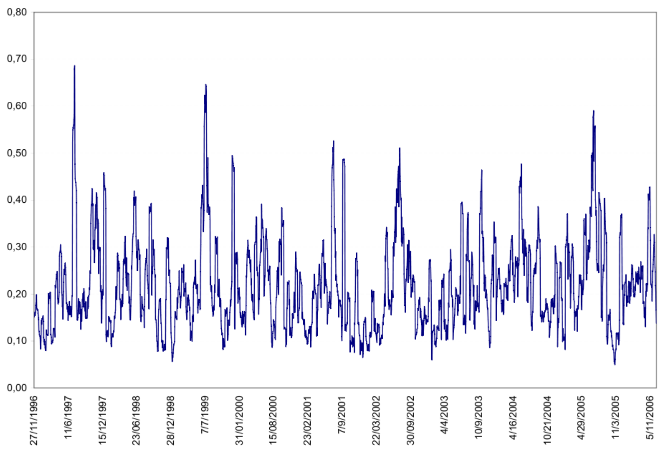


(c) Soya convenience yield.

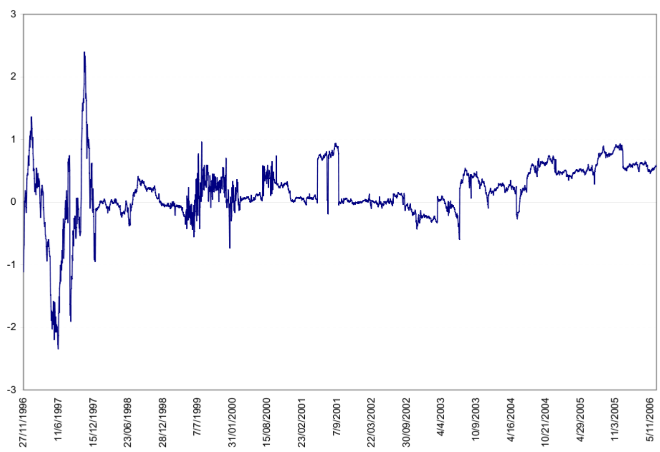
Figure 2. Soya prices, volatility and convenience yield, from November, 1996 to March, 2006.



(a) Corn prices.



(b) Corn volatility.



(c) Corn convenience yield.

Figure 3. Corn prices, volatility and convenience yield, from November, 1996 to March, 2006.

Coefficients	**Wheat** GBM	**Wheat** IHGBM	**Soya** GBM	**Soya** IHGBM	**Corn** GBM	**Corn** IHGBM
a		0.9869		0.8666		0.7101
		(3.63)		(1.35)		(2.98)
b	0.000054	-0.00862	0.000304	-0.005809	-0.000045	-0.007936
	(0.16)	(-3.60)	(1.00)	(-1.65)	(-0.15)	(-2.97)
σ_1^2	0.000268	0.000265	0.000198	0.000255	0.000219	0.000231
	(17.54)	(17.30)	(10.54)	(5.57)	(15.30)	(15.62)
γ	1	1	1	1	1	1
Speed of adjustment λ		0.0086		0.0058		0.0079
Long-run mean $\overline{X}$		114.48		149.23		89.48
Volatility σ_1	0.0164	0.0163	0.0141	0.0160	0.0148	0.0152

Table 1. GMM estimation for the Corn, Soya and Wheat prices. This Table includes the estimated parameters (with t-values in parentheses) for the one-factor models proposed for crop prices. The sample period covers from November, 1996 to March, 2006 for a total of 2,363 daily observations. The weighting matrix is based on Newey and West (1987) with 12 lags and with initial weighting matrix given by the instrumental variables to ensure convergence.

We now turn to the valuation of the put option embedded in the land. To do so, we simulate $10{,}000$ crop price paths for each process. Considering an interest rate of 7 percent, we then calculate the value of the discounted cash flows to the land owner assuming that he exploits the land forever. We focus on land prices in Argentina. As these prices are given in USD per Ha, we must use some measure of soil productivity to transform the simulated crop prices to USD per Ha. We assume that land productivity is constant. In more detail, we will use an average of the productivity of farm land in Argentina.

Average Productivity of the soil in Argentina

Crop	Tons per Ha
Wheat	2
Soya	2.8
Corn	3

Source: Márgenes Agropecuarios.

The results, in U.S. dollars per Ha, are the following:

	Wheat Put	**Wheat** Land Value	**Soya** Put	**Soya** Land Value	**Corn** Put	**Corn** Land Value
One-factor models						
GBM	$1{,}668$	$5{,}700$	$3{,}857$	$13{,}168$	$1{,}566$	$6{,}063$
IHGBM	$1{,}662$	$5{,}686$	$3{,}772$	$12{,}902$	$1{,}758$	$5{,}985$
Two-factor models						
GBM + O-U	528	$1{,}828$	442	$1{,}565$	510	$1{,}785$
GBM + IHGBM	38	126	76	235	54	183

As we can see, the value of the put option is by no means insignificant. The value of being the owner has some strategic content which is, at least, 25.8 percent of the total land value. For each crop, one-factor models provide very similar results. For two-factor models, when the convenience yield follows an O-U process, put prices and land values are much higher than when using a IHGBM process. The reason could be that the IHGBM forces the convenience yield to be always positive which does not seem to be realistic.

Coefficients	Wheat		Soya		Corn	
	IHGBM	O-U	IHGBM	O-U	IHGBM	O-U
a	0.0076	0.0098	0.0024	0.0027	0.0048	0.00656
	(4.05)	(4.40)	(1.64)	(1.57)	(2.16)	(2.71)
b	−0.0301	−0.0380	−0.0418	−0.0454	−0.0178	−0.0233
	(−4.20)	(−4.50)	(−2.83)	(−2.99)	(−2.79)	(−3.29)
σ_2^2	0.002101	0.000265	0.002	0.000255	0.014351	0.002716
	(4.65)	(5.20)	(4.57)	(2.83)	(5.10)	(6.71)
γ	1	0	1	0	1	0
Speed of adjustment κ	0.0301	0.0380	0.0418	0.0454	0.0178	0.0233
Long-run mean α	0.2527%	0.2578%	0.058%	0.0597%	0.2666%	0.2809%
Volatility σ_2	0.0458	0.1628	0.0447	0.1597	0.1198	0.0521

Table 2. GMM estimation for the Corn, Soya and Wheat convenience yields. This Table includes the estimated parameters (with t-values in parentheses) for the processes proposed for the convenience yields. The sample period covers from November, 1996 to March, 2006 for a total of 2,363 daily observations. The weighting matrix is based on Newey and West (1987) with 12 lags and with initial weighting matrix given by the instrumental variables to ensure convergence.

We can also see that one-factor processes provide higher prices than two-factor models. This may happen because the (stochastic) convenience yield can reduce the level of crop prices over the long run.[8]

Are our calculated prices close to the market prices of agricultural land? The following Table reports the land prices in Argentina for the period 1997–2006.

Agricultural Land Prices in Argentina										
Year	1997	1998	1999	2000	2001	2002	2003	2004	2005	2006
USD per Ha	4,042	4,858	4,000	3,950	3,592	3,000	3,900	5,360	6,100	7,500

Source: Márgenes Agropecuarios.

Comparing these prices to our theoretical prices, we see that one-factor models undervalue the land when seeded with wheat and corn and overvalue it for the case of soya. Obviously, from the land owner point of view, it is optimal to seed the land with soya, since it leads to the highest land and put values. However, seeding always the same crop can deteriorate the land. The farmer should take into account this effect rotating the crops according to a certain rule. This issue complicates the current analysis and it is left for further research.

5 Conclusions

The price of an asset should be tied to its dividends, payoffs or cash-flows. Agricultural lands are in no sense different to this. We have proposed different processes to compute the cash-flows from the land. We assume that the land can be seeded with three crops: wheat, soya, and corn. We have considered one-factor models in which the crop prices are given by either GBM or IHGBM, respectively. We have also used a couple of two-factor models where the convenience yield is stochastic. In one case, it follows a O-U process and, in another case, it follows an IHGBM process. The corresponding parameters have been estimated by applying the GMM technique. With these estimates, we value the put option embedded in the land price. Our application focuses on Argentina and the findings show that this option accounts for, at least, one quarter of the land value. We have also found that all the models undervalue the land for wheat and corn and overvalue it for soya.

[8] As can be seen in the Table containing the descriptive statistics, all the convenience yields have a positive mean value.

For further research, several variables can be incorporated into the analysis. For instance, we can mention three of them as the most relevant ones: weather conditions, variable soil productivity and governments tax policies. Additionally, crop rotation can be included in the analysis so that the optimal seeding policy is determined.

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A Appendix

- The first moment for the Inhomogeneous Geometric Brownian Motion process (1) can be computed in the following way:

$$E(\mathrm{d}X_t) = \lambda(\overline{X} - E(X_t))\,\mathrm{d}t$$

Using the linearity of the expectation, we arrive at the following formula

$$\frac{\mathrm{d}E(X_t)}{\mathrm{d}t} = \lambda(\overline{X} - E(X_t))$$

Then,

$$\mathrm{e}^{\lambda t}\left[\frac{\mathrm{d}E(X_t)}{\mathrm{d}t} + \lambda E(X_t)\right] = \mathrm{e}^{\lambda t}\lambda\overline{X}$$

Integrating this equation, we can find the first moment:

$$\int_0^t e^{\lambda t}\left[\frac{dE(X_t)}{dt}+\lambda E(X_t)\right]dt=\int_0^t e^{\lambda t}\lambda\overline{X}\,dt$$

Then, we get

$$E(X_t)e^{\lambda t}-E\left[X_0\right]=\overline{X}e^{\lambda t}-\overline{X}$$

or, equivalently

$$E(X_t)=\overline{X}+(X_0-\overline{X})e^{-\lambda t}$$

As expected, as time goes to infinity, $E(X_\infty)=\overline{X}$, corroborating that $\overline{X}$ is the long-term value at which the variable X converges to.

- For the second moment, defining $f(X_t)=X_t^2$ and applying Itô's lemma, we obtain

$$\frac{dE(X_t^2)}{dt}=2\lambda\overline{X}E(X_t)+(\sigma^2-2\lambda)E(X_t^2)$$

Substituting for $E(X_t)$, we get

$$\frac{dE(X_t^2)}{dt}+(2\lambda-\sigma^2)E(X_t^2)=2\lambda\overline{X}\left(\overline{X}+(X_0-\overline{X})e^{-\lambda t}\right)$$

Using $(2\lambda-\sigma^2)t$ as the integrating factor, we have

$$\int_0^t e^{(2\lambda-\sigma^2)t}\left[\frac{dE(X_t^2)}{dt}+(2\lambda-\sigma^2)E(X_t^2)\right]dt=\int_0^t e^{(2\lambda-\sigma^2)t}[2\lambda\overline{X}(\overline{X}+(X_0-\overline{X})e^{-\lambda t})]\,dt$$

Solving the integral, we obtain

$$E(X_t^2)e^{(2\lambda-\sigma^2)t}-X_0^2=\int_0^t 2\lambda\overline{X}^2e^{(2\lambda-\sigma^2)t}\,dt+\int_0^t 2\lambda\overline{X}\left(X_0-\overline{X}e^{(\lambda-\sigma^2)t}\right)dt$$

Then,

$$e^{(2\lambda-\sigma^2)t}E(X_t^2)=\frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2}(e^{(2\lambda-\sigma^2)t}-1)+\frac{2\lambda\overline{X}(X_0-\overline{X})}{\lambda-\sigma^2}(e^{(\lambda-\sigma^2)t}-1)+X_0^2$$

Provided that $(2\lambda-\sigma^2)(\lambda-\sigma^2)\neq 0$, we obtain[9]

$$E(X_t^2)=\frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2}\left(1-e^{(\sigma^2-2\lambda)t}\right)+\frac{2\lambda\overline{X}(X_0-\overline{X})}{\lambda-\sigma^2}\left(e^{-\lambda t}-e^{(\sigma^2-2\lambda)t}\right)+X_0^2e^{(\sigma^2-2\lambda)t}$$

As $\mathrm{Var}(X_t^2)=E[(X_t-E(X_t))^2]$, the second non-central moment can be computed as

$$\begin{aligned}\mathrm{Var}(X_t)=e^{(\sigma^2-2\lambda)t}&\left(X_0^2+\frac{2\lambda\overline{X}^2}{\sigma^2-2\lambda}+\frac{2\lambda\overline{X}(X_0-\overline{X})}{\sigma^2-\lambda}\right)\\&+e^{-\lambda t}\left(\frac{2\lambda\overline{X}(X_0-\overline{X})}{\lambda-\sigma^2}-2\overline{X}(X_0-\overline{X})\right)\\&-e^{-2\lambda t}(X_0-\overline{X})^2+\frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2}-\overline{X}^2\end{aligned}$$

[9] We actually have two more possibilities: If $\sigma^2-2\lambda=0$, then $E(X_t^2)=-(2k\overline{X}^2)t+2(X_0-\overline{X})(e^{-\lambda t}-1)+X_0^2$. Alternatively, if $\lambda=\sigma^2$, then $E(X_t^2)=2\overline{X}^2(1-e^{-\lambda t})+2\overline{X}(\overline{X}-X_0)t\,e^{-\lambda t}+X_0^2e^{-\lambda t}$.

So $\mathrm{Var}(X_\infty) = \frac{2\lambda\overline{X}^2}{2\lambda-\sigma^2} - \overline{X}^2 = \frac{\sigma^2}{2\lambda-\sigma^2}\overline{X}^2$. Then, this variance converges to a finite value if and only if the speed of mean reversion is high enough $(\lambda > \sigma^2/2)$.

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