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# Recent progress and open questions on the numerical index of Banach spaces

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**Abstract** The aim of this paper is to review the state-of-the-art of recent research concerning the numerical index of Banach spaces, by presenting some of the results found in the last years and proposing a number of related open problems.

**Resultados recientes y problemas abiertos sobre el índice numérico de los espacios de Banach**

**Resumen.** El propósito de este trabajo es revisar el estado actual de la investigación reciente sobre el índice numérico de los espacios de Banach, presentando algunos de los resultados obtenidos en los últimos años y proponiendo un cierto número de problemas abiertos.

## Introduction

The numerical index of a Banach space is a constant relating the behavior of the numerical range with that of the usual norm on the Banach algebra of all bounded linear operators on the space. The notion of numerical range (also called field of values) was first introduced by O. Toeplitz in 1918 [79] for matrices, but his definition applies equally well to operators on infinite-dimensional Hilbert spaces. If $H$ denotes a Hilbert space with inner product $(\cdot \mid \cdot)$, the *numerical range* of a bounded linear operator $T$ on $H$ is the subset $W(T)$ of the scalar field defined by

$$W(T) := \big\{(Tx \mid x) \ : \ x \in H, \ (x \mid x) = 1\big\}.$$

Some properties of the Hilbert space numerical range are discussed in the classical book of P. Halmos [29, §17]. Let us just mention that the numerical range of a bounded linear operator is (surprisingly) convex and, in the complex case, its closure contains the spectrum of the operator. Moreover, if the operator is normal, then the closure of its numerical range coincides with the convex hull of its spectrum. Further developments can be found in a recent book of K. Gustafson and D. Rao [28]. In the sixties, the concept of numerical range was extended to operators on general Banach spaces by G. Lumer [54] and F. Bauer [4]. Let us give the necessary definitions. Given a real or complex Banach space $X$, we write $B_X$ for the closed unit ball and $S_X$ for the unit sphere of $X$. The dual space will be denoted by $X^*$ and $L(X)$ will be the Banach algebra of all bounded linear operators on $X$. The *numerical range* of an operator $T \in L(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) := \{x^*(Tx) \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\}.$$

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The *numerical radius* is the seminorm defined on $L(X)$ by

$$v(T) := \sup\{|\lambda| \ : \ \lambda \in V(T)\}$$

for $T \in L(X)$. Classical references here are the monographs by F. Bonsall and J. Duncan [9, 10] from the seventies. Let us mention that the numerical range of a bounded linear operator is connected (but not necessarily convex, see [10, Example 21.6]) and, in the complex case, its closure contains the spectrum of the operator. The theory of numerical ranges has played a crucial role in the study of some algebraic structures, especially in the non-associative context (see the expository paper [46] by A. Kaidi, A. Morales, and A. Rodríguez Palacios, for example).

The concept of *numerical index* of a Banach space $X$ was first suggested by G. Lumer in 1968 (see [20]), and it is the constant $n(X)$ defined by

$$n(X) := \inf\{v(T) \ : \ T \in L(X), \ \|T\| = 1\}$$

or, equivalently,

$$n(X) = \max\{k \geqslant 0 : \ k\,\|T\| \leqslant v(T) \ \ \forall\, T \in L(X)\}.$$

Note that $n(X) > 0$ if and only if $v$ and $\|\cdot\|$ are equivalent norms on $L(X)$. At that time, it was known that in a complex Hilbert space $H$ with dimension greater than 1, $\|T\| \leqslant 2v(T)$ for all $T \in L(H)$, and 2 is the best constant; in the real case, there exists a norm-one operator whose numerical range reduces to zero. In our terminology, $n(H) = 1/2$ if $H$ is complex, and $n(H) = 0$ if it is real. Actually, real and complex general Banach spaces behave in a very different way with regard to the numerical index, as summarized in the following equalities by J. Duncan, C. McGregor, J. Pryce, and A. White [20]:

$$\begin{aligned} \{n(X) \ : \ X \text{ complex Banach space }\} &= [\mathrm{e}^{-1}, 1], \qquad (1)\\ \{n(X) \ : \ X \text{ real Banach space }\} &= [0, 1]. \end{aligned}$$

The fact that $n(X) \geqslant \mathrm{e}^{-1}$ for every complex Banach space $X$ was observed by B. Glickfeld [26] (by making use of a classical theorem of H. Bohnenblust and S. Karlin [7]), who also gave an example where this inequality becomes an equality. It is showed in the already cited paper [20], that $M$-spaces, $L$-spaces and their isometric preduals, have numerical index 1, a property shared by the disk algebra (M. Crab, J. Duncan, and C. McGregor [16, Theorem 3.3]). Finally, let us mention that the real space $X_{\mathbb{R}}$ underlying a complex Banach space $X$ satisfies $n(X_{\mathbb{R}}) = 0$; actually, the isometry $x \longmapsto i\,x$ has numerical radius 0 when viewed as an operator on $X_{\mathbb{R}}$.

In the last ten years, many results on the numerical index of Banach spaces have appeared in the literature. This paper aims at reviewing the state of the art on this topic and proposing a variety of open questions. The structure of our discussion is as follows.

# Contents

We finish this introduction by recalling some definitions and fixing notation. We write $\mathbb{T}$ to denote the unit sphere of the base field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). We use the notation $\operatorname{Re}(\,\cdot\,)$ to denote the real part function, which should be considered as the identity when $\mathbb{K} = \mathbb{R}$. Given a real or complex Banach space $X$, we write $\operatorname{co}(B)$ and $\overline{\operatorname{co}}(B)$ to denote, respectively, the convex hull and the closed convex hull of a set $B \subseteq X$ and we denote by $\operatorname{ext}(C)$ the set of extreme points of a convex set $C \subseteq X$. A subset $A$ of $B_{X^*}$ is said to be *norming (for $X$)* if

$$\|x\| = \sup\{|x^*(x)| \ : \ x^* \in A\} \qquad \big(x \in X\big)$$

or, equivalently, if $B_{X^*} = \overline{\operatorname{co}}^{w^*}(\mathbb{T}\,A)$ (Hahn-Banach Theorem). Finally, for $1 \leqslant p \leqslant \infty$, we write $\ell_p^m$ to denote the normed space $\mathbb{K}^m$ endowed with the usual $p$-norm, and we write $X \oplus_p Y$ to denote the $\ell_p$-direct sum of the spaces $X$ and $Y$.

# 1. Computing the numerical index

In view of the examples given in the introduction, the most important family of classical Banach spaces (in the sense of H. Lacey [47]) whose numerical indices remain unknown is the family of $L_p$ spaces when $p \neq 1, 2, \infty$. This is actually one of the most intriguing open problems in the field but, very recently, E. Ed-Dari and M. Khamsi [21, 22] have made some progress. We summarize their results in the following statement and use it to motivate some conjectures.

**Theorem 1 (**[21, 22]**)** *Let $1 \leqslant p \leqslant \infty$ be fixed. Then,*

*(a) $n(L_p[0,1]) = n(\ell_p) = \inf\{n(\ell_p^m) \ : \ m \in \mathbb{N}\}$, and the sequence $\{n(\ell_p^m)\}_{m\in\mathbb{N}}$ is decreasing.*

*(b) $n(L_p(\mu)) \geqslant n(\ell_p)$ for every positive measure $\mu$.*

*(c) In the real case,*

$$\frac{1}{2} M_p \leqslant n(\ell_p^2) \leqslant M_p, \qquad \textit{where } \ M_p = \sup_{t\in[0,1]} \frac{|t^{p-1} - t|}{1 + t^p}.$$

When $p \neq 2$, it is known that $n(\ell_p^m) > 0$ for every $m \geqslant 2$, and also that $v(T) > 0$ for every non-null $T \in L(\ell_p)$ (see [22, Theorem 2.3 and subsequent remark]), but we do not know if $n(\ell_p) > 0$. Observe that a positive answer to this question implies, thanks to (b) above, that $n(L_p(\mu)) > 0$ for every positive measure $\mu$.

**Problem 1** *Is $n(\ell_p)$ positive for every $p \neq 2$?*

With respect to item (c) in the above theorem, let us explain the meaning of the number $M_p$. It can be deduced from [20, §3] that, given an operator $T \in L(\ell_p^2)$ represented by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, one has

$$v(T) = \max\left\{ \max_{\substack{t\in[0,1]\\ z\in\mathbb{T}}} \frac{\big|a + d\,t^p + z\,b\,t + \overline{z}\,c\,t^{p-1}\big|}{1 + t^p},\ \max_{\substack{t\in[0,1]\\ z\in\mathbb{T}}} \frac{\big|d + a\,t^p + \overline{z}\,c\,t + z\,b\,t^{p-1}\big|}{1 + t^p} \right\}. \qquad (2)$$

It follows that $M_p$ is equal to the numerical radius of the norm-one operator $U \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in $L(\ell_p^2)$ (real case), so $n(\ell_p^2) \leqslant M_p$. For $p = 2$, the operator $U$ has minimum numerical radius, namely $0$. We may ask if $U$ is also the norm-one operator with minimum numerical radius for all the real spaces $\ell_p^2$.

**Problem 2** *Is it true that, in the real case,* $n(\ell_p^2) = \sup\limits_{t\in[0,1]} \dfrac{|t^{p-1}-t|}{1+t^p}$ *for every* $1<p<\infty$*?*

In the complex case, the operator $U$ acting on $\ell_2^2$ satisfies $v(U) = \|U\|$ (take $z=i$ and $t=1$ in (2)) and, therefore, its numerical radius is not the minimum. Actually, one has

$$n(\ell_2^2) = \frac{1}{2} = v(S),$$

where $S \in L(\ell_2^2)$ is the 'shift' $S \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Therefore, we bet that $n(\ell_p^2) = v(S)$ for every $p$ in the complex case. It can be checked from (2) that

$$v(S) = \frac{(p-1)^{\frac{p-1}{p}}}{p} = \frac{1}{p^{\frac{1}{p}}\, q^{\frac{1}{q}}},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

**Problem 3** *Is it true that, in the complex case,* $n(\ell_p^2) = \dfrac{1}{p^{\frac{1}{p}}\, q^{\frac{1}{q}}}$ *for every* $1<p<\infty$*?*

In view of Theorem 1.a, the two-dimensional case is only the first step in the computation of $n(\ell_p)$, but it is reasonable to expect that the sequence $\{n(\ell_p^m)\}_{m\in\mathbb{N}}$ is always constant, as it happens in the cases $p=1,2,\infty$.

**Problem 4** *Is it true that* $n(\ell_p) = n(\ell_p^2)$ *for every* $1<p<\infty$*?*

In a 1977 paper [35], T. Huruya determined the numerical index of a $C^*$-algebra. Part of the proof was recently clarified by A. Kaidi, A. Morales, and A. Rodríguez-Palacios in [45], where the result is extended to $JB^*$-algebras and preduals of $JBW^*$-algebras. Let us state here those results just for $C^*$-algebras and preduals of von Neumann algebras.

**Theorem 2 (**[35] and [45, Proposition 2.8]**)** *Let $A$ be a $C^*$-algebra. Then, $n(A)$ is equal to $1$ or $\frac{1}{2}$ depending on whether $A$ is commutative or not. If $A$ is actually a von Neumann algebra with predual $A_*$, then $n(A_*) = n(A)$.*

We do not know if there is an analogous result in the real case. We recall that a *real $C^*$-algebra* can be defined as a norm-closed self-adjoint real subalgebra of a complex $C^*$-algebra, and a *real $W^*$-algebra* (or *real von Neumann algebra*) is a real $C^*$-algebra which admits a predual (see [36] for more information).

**Problem 5** *Compute the numerical index of real $C^*$-algebras and isometric preduals of real $W^*$-algebras.*

The fact that the disk algebra has numerical index 1 was extended to function algebras by D. Werner in 1997 [82]. A *function algebra $A$* on a compact Hausdorff space $K$ is a closed subalgebra of $C(K)$ which separates the points of $K$ and contains the constant functions.

**Proposition 1 (**[82, Corollary 2.2 and proof of Theorem 3.3]**)** *If $A$ is a function algebra, then $n(A) = 1$.*

Of course, there are many other Banach spaces whose numerical index is unknown. We propose to calculate some of them.

**Problem 6** *Compute the numerical index of $C^m[0,1]$ (the space of $m$-times continuously differentiable real functions on $[0,1]$, endowed with any of its usual norms), Lip$(K)$ (the space of all Lipschitz functions on the complete metric space $K$), Lorentz spaces, and Orlicz spaces.*

Some of the classical results given in the introduction about the numerical index of particular spaces have been extended to sums of families of Banach spaces and to spaces of vector-valued functions in various papers by G. López, M. Martín, J. Merí, R. Payá, and A. Villena [52, 62, 64].

We start by presenting the result for sums of spaces. Given an arbitrary family $\{X_\lambda \,:\, \lambda \in \Lambda\}$ of Banach spaces, we denote by $[\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}$, $[\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}$ and $[\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}$ the $c_0$-, $\ell_1$- and $\ell_\infty$-sum of the family.

**Proposition 2 (**[62, Proposition 1]**)** *Let $\{X_\lambda \,:\, \lambda \in \Lambda\}$ be a family of Banach spaces. Then*

$$n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{c_0}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_1}\Big) = n\Big([\oplus_{\lambda\in\Lambda} X_\lambda]_{\ell_\infty}\Big) = \inf_\lambda n(X_\lambda).$$

The above result is not true for $\ell_p$-sums if $p$ is different from 1 and $\infty$. Nevertheless, it is possible to give one inequality and, actually, the same is true for absolute sums [56]. Recall that a direct sum $Y \oplus Z$ is said to be an *absolute sum* if $\|y+z\|$ only depends on $\|y\|$ and $\|z\|$ for $(y,z) \in Y \times Z$. For background on absolute sums the reader is referred to [66] and references therein.

**Proposition 3 (**[56, Proposición 1]**)** *Let $X$ be a Banach space and let $Y$, $Z$ be closed subspaces of $X$. Suppose that $X$ is the absolute sum of $Y$ and $Z$. Then*

$$n(X) \leqslant \min\big\{n(Y),\ n(Z)\big\}.$$

The following somehow surprising example was obtained in [62] by using Proposition 2.

**Example 1 (**[62, Example 2.b]**)** *There is a real Banach space $X$ for which the numerical radius is a norm but is not equivalent to the operator norm, i.e. the numerical index of $X$ is 0 although $v(T) > 0$ for every non-null $T \in L(X)$.*

The numerical index of some vector-valued function spaces was also computed in [52, 62, 64]. Given a real or complex Banach space $X$ and a compact Hausdorff topological space $K$, we write $C(K,X)$ and $C_w(K,X)$ to denote, respectively, the space of $X$-valued continuous (resp. weakly continuous) functions on $K$. If $\mu$ is a positive $\sigma$-finite measure, by $L_1(\mu,X)$ and $L_\infty(\mu,X)$ we denote respectively the space of $X$-valued $\mu$-Bochner-integrable functions and the space of $X$-valued $\mu$-Bochner-measurable functions which are essentially bounded.

**Theorem 3 (**[52], [62], and [64]**)** *Let $K$ be a compact Hausdorff space, and let $\mu$ be a positive $\sigma$-finite measure. Then*

$$n(C_w(K,X)) = n(C(K,X)) = n(L_1(\mu,X)) = n(L_\infty(\mu,X)) = n(X)$$

*for every Banach space $X$.*

The numerical index of $C_{w^*}(K,X^*)$, the space of $X^*$-valued weakly-star continuous functions on $K$ is also studied in [52]. Unfortunately, only a partial result is achieved.

**Proposition 4 (**[52, Propositions 5 and 7] **)** *Let $K$ be a compact Hausdorff space and let $X$ be a Banach space. Then*

$$n(C_{w^*}(K,X^*)) \leqslant n(X).$$

*If, in addition, $X$ is an Asplund space or $K$ has a dense subset of isolated points, then*

$$n(X^*) \leqslant n(C_{w^*}(K,X^*)).$$

To finish this section let us comment that, roughly speaking, when one finds an explicit computation of the numerical index of a Banach space in the literature only few values appear; namely, 0 (real Hilbert spaces), $\mathrm{e}^{-1}$ (Glickfeld's example), $1/2$ (complex Hilbert spaces), and 1 ($C(K)$, $L_1(\mu)$, and many more). The preceding results about sums and vector-valued function spaces do not help so much, and the exact values of $n(\ell_p^2)$ are not still known. Let us also say that, when the authors of [20] prove (1), they only use examples of Banach spaces whose numerical indices are the extremes of the intervals, and then a connectedness argument is applied. Very recently, M. Martín and J. Merí have partially covered this gap in [59], where they explicitly compute the numerical index for four families of norms on $\mathbb{R}^2$. The most interesting one is the family of regular polygons.

**Proposition 5 (**[59, Theorem 5]**)** *Let $n \geqslant 2$ be a positive integer, and let $X_n$ be the two-dimensional real normed space whose unit ball is the convex hull of the $(2n)^{\mathrm{th}}$ roots of unity, i.e. $B_{X_n}$ is a regular $2n$-polygon centered at the origin and such that one of its vertices is $(1,0)$. Then,*

$$n(X_n) = \begin{cases} \tan\left(\dfrac{\pi}{2n}\right) & \text{if } n \text{ is even,} \\[2ex] \sin\left(\dfrac{\pi}{2n}\right) & \text{if } n \text{ is odd.} \end{cases}$$

**Problem 7** *Compute the numerical index for other families of finite-dimensional Banach spaces. In particular, it would be interesting to get the complex analog of the results in* [59].

## 2. Numerical index and duality

Given a bounded linear operator $T$ on a Banach space $X$, it is a well-known fact in the theory of numerical ranges (see [9, §9]) that

$$\sup \operatorname{Re} V(T) = \lim_{\alpha \downarrow 0} \frac{\|\mathrm{Id} + \alpha\, T\| - 1}{\alpha} \tag{3}$$

and so

$$v(T) = \max_{\omega \in \mathbb{T}} \lim_{\alpha \downarrow 0} \frac{\|\mathrm{Id} + \alpha\, \omega\, T\| - 1}{\alpha}.$$

It is immediate from the above formula that

$$v(T) = v(T^*), \tag{4}$$

where $T^* \in L(X^*)$ is the adjoint operator of $T$, and the result given in [20, Proposition 1.3] that

$$n(X^*) \leqslant n(X) \tag{5}$$

clearly follows. The question if this is actually an equality had been around from the beginning of the subject (see [46, pp. 386], for instance). Let us comment some partial results which led to think that the answer could be positive. Namely, it is clear that $n(X) = n(X^*)$ for every reflexive space $X$, and this equality also holds whenever $n(X^*) = 1$ , in particular when $X$ is an $L$- or an $M$-space. Moreover, it is also true that $n(X) = n(X^*)$ when $X$ is a $C^*$-algebra or a von Neumann algebra predual (Theorem 2).

Nevertheless, in a very recent paper [12], K. Boyko, V. Kadets, M. Martín, and D. Werner have answered the question in the negative by giving an example of a Banach space whose numerical index is strictly greater than the numerical index of its dual. Our aim in this section is to present such counterexample with a new and more direct proof.

Let us recall that $c$ denotes the Banach space of all convergent scalar sequences $x = (x(1), x(2), \ldots)$ equipped with the sup-norm. The dual space of $c$ is (isometric to) $\ell_1$ and we will write $c^* \equiv \ell_1 \oplus_1 \mathbb{K}$ where

$$\langle (y,\lambda)\,,\, x\rangle = \sum_{n=1}^{\infty} y(n)\, x(n) \;+\; \lambda \lim x \qquad \big(x \in c,\ (y,\lambda) \in \ell_1 \oplus_1 \mathbb{K}\big).$$

For every $n \in \mathbb{N}$, we denote by $e_n^*$ the norm-one element of $c^*$ given by

$$e_n^*(x) = x(n) \qquad \big(x \in c\big).$$

We are now ready to show that the numerical index of a Banach space and the one of its dual do not always coincide.

**Example 2 (**[12, Example 3.1]**)** *Let us consider the Banach space*

$$X = \big\{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\big\}.$$

*Then,* $n(X) = 1$ *and* $n(X^*) < 1$.

PROOF. We observe that

$$X^* = \big[c^* \oplus_1 c^* \oplus_1 c^*\big]/\langle(\lim, \lim, \lim)\rangle$$

so that, writing $Z = \ell_1^3/\langle(1, 1, 1)\rangle$, we can identify

$$X^* \equiv \ell_1 \oplus_1 \ell_1 \oplus_1 \ell_1 \oplus_1 Z \qquad \text{and} \qquad X^{**} \equiv \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty \ell_\infty \oplus_\infty Z^*. \tag{6}$$

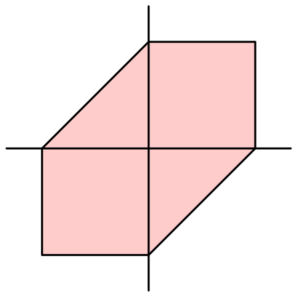

Figure 1. The unit ball of $Z$

With this in mind, we write $A$ to denote the set

$$\{(e_n^*, 0, 0, 0) \ : \ n \in \mathbb{N}\} \cup \{(0, e_n^*, 0, 0) \ : \ n \in \mathbb{N}\} \cup \{(0, 0, e_n^*, 0) \ : \ n \in \mathbb{N}\}.$$

Then $A$ is clearly a norming subset of $S_{X^*}$ and

$$|x^{**}(a^*)| = 1 \qquad \big(x^{**} \in \text{ext}(B_{X^{**}}),\ a^* \in A\big). \tag{7}$$

Let us prove that $n(X) = 1$. Indeed, we fix $T \in L(X)$ and $\varepsilon > 0$. Since $T^*$ is $w^*$-continuous and $A$ is norming, we may find $a^* \in A$ such that

$$\|T^*(a^*)\| \geqslant \|T\| - \varepsilon.$$

Now, we take $x^{**} \in \text{ext}(B_{X^{**}})$ such that

$$|x^{**}(T^*(a^*))| = \|T^*(a^*)\|.$$

Since $|x^{**}(a^*)| = 1$ thanks to (7), we get

$$v(T) = v(T^*) \geqslant |x^{**}(T^*(a^*))| \geqslant \|T\| - \varepsilon.$$

It clearly follows that $v(T) = \|T\|$ and $n(X) = 1$.

To show that $n(X^*) < 1$, we use (6) and Proposition 2 to get

$$n(X^*) \leqslant n(Z),$$

and the fact that $n(Z) < 1$ follows easily from a result due to C. McGregor [65, Theorem 3.1]. Actually, in the real case, the unit ball of $Z$ is an hexagon (see Figure 1 above), which is isometrically isomorphic to the space $X_3$ of Proposition 5, so $n(Z) = 1/2$. ■

The above example can be pushed forward, to produce even more striking counterexamples.

**Examples 3 (**[12, Examples 3.3]**)**

*(a) There exists a real Banach space $X$ such that $n(X) = 1$ and $n(X^*) = 0$.*

*(b) There exists a complex Banach space $X$ satisfying that $n(X) = 1$ and $n(X^*) = 1/\mathrm{e}$.*

Once we know that the numerical index of a Banach space and the one of its dual may be different, the question aries if two preduals of a given Banach space have the same numerical index. By a *predual* of a Banach space $Y$ we mean a Banach space $X$ such that $X^*$ is (isometrically isomorphic to) $Y$. The answer is again negative, as the following result shows.

**Example 4 (**[12, Example 3.6]**)** *Let us consider the Banach spaces*

$$X_1 = \big\{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ \lim x + \lim y + \lim z = 0\big\}$$

*and*

$$X_2 = \big\{(x, y, z) \in c \oplus_\infty c \oplus_\infty c \ : \ x(1) + y(1) + z(1) = 0\big\}.$$

*Then, $X_1^*$ and $X_2^*$ are isometrically isomorphic, but $n(X_1) = 1$ and $n(X_2) < 1$.*

The following question might also be addressed.

**Problem 8** *Let $Y$ be a dual space. Does there exist a predual $X$ of $Y$ such that $n(X) = n(Y)$?*

Another interesting issue could be to find isomorphic properties of a Banach space $X$ ensuring that $n(X^*) = n(X)$. On the one hand, Example 2 shows that Asplundness is not such a property. On the other hand, it is shown in [12, Proposition 4.1] that if a Banach space $X$ with the Radon-Nikodým property has numerical index 1, then $X^*$ has numerical index 1 as well. Therefore, the following question naturally arises.

**Problem 9** *Let $X$ be a Banach space with the Radon-Nikodým property. Is it true that $n(X) = n(X^*)$?*

Another sufficient condition would follow from a positive answer to Problem 8.

**Problem 10** *Let $Y$ be a dual space admitting a unique predual $X$ (up to isometric isomorphisms). Is it true that $n(Y) = n(X)$?*

Let us finish this section by remarking that the space given in Example 2 is useful as a counterexample for many other conjectures, as we will see later on.

# 3. Banach spaces with numerical index one

The guiding open question on these spaces is the following.

**Problem 11** *Find necessary and sufficient conditions for a Banach space to have numerical index* $1$ *which do not involve operators.*

In 1971, C. McGregor [65, Theorem 3.1] gave such a characterization in the finite-dimensional case. More concretely, a finite-dimensional normed space $X$ has numerical index 1 if and only if

$$|x^*(x)| = 1 \quad \text{for every } x \in \operatorname{ext}(B_X) \text{ and every } x^* \in \operatorname{ext}(B_{X^*}). \tag{8}$$

It is not clear how to extend this result to arbitrary Banach spaces. If we use literally (8) in the infinite-dimensional context, we do not get a sufficient condition, since the set $\operatorname{ext}(B_X)$ may be empty and this does not imply numerical index 1 (e.g. $\operatorname{ext}(B_{c_0(\ell_2)}) = \emptyset$ but $n(c_0(\ell_2)) < 1$). On the other hand, we do not know if (8) is a necessary condition.

**Problem 12** *Let* $X$ *be a Banach space with numerical index* $1$*. Is it true that* $|x^*(x)| = 1$ *for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every* $x \in \operatorname{ext}(B_X)$*?*

Our first aim in this section is to discuss several reformulations of assertion (8) to get either sufficient or necessary conditions for a Banach space to have numerical index $1$.

Aiming at sufficient conditions, it is not difficult to show that (8) implies numerical index 1 for a Banach space $X$ as soon as the set $\operatorname{ext}(B_X)$ is large enough to determine the norm of operators on $X$, i.e. $B_X = \overline{\operatorname{co}}(\operatorname{ext}(B_X))$. Actually, we may replace $\operatorname{ext}(B_X)$ with any subset of $S_X$ satisfying the same property. On the other hand, we may replace $\operatorname{ext}(B_X)$ by $\operatorname{ext}(B_{X^{**}})$ and the role of $\operatorname{ext}(B_{X^*})$ can be played by any norming subset of $S_{X^*}$. Let us comment that this is is what we did in the proof of Example 2. All these ideas appear implicity in several papers (see [53, 57, 58] for example); we summarize them in the following proposition.

**Proposition 6** *Let* $X$ *be a Banach space. Then, any of the following three conditions is sufficient to ensure that* $n(X) = 1$.

*(a) There exists a subset* $C$ *of* $S_X$ *such that* $\overline{\operatorname{co}}(C) = B_X$ *and*

$$|x^*(c)| = 1$$

*for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every* $c \in C$.

*(b)* $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every* $x^* \in \operatorname{ext}(B_{X^*})$.

*(c) There exists a norming subset* $A$ *of* $S_{X^*}$ *such that*

$$|x^{**}(a^*)| = 1$$

*for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every* $a^* \in A$.

Let us comment on the converse of the above result. First, condition (a) is not necessary as shown by $c_0$. Second, it was proved in [12, Example 3.4] that condition (b) is not necessary either, the counterexample being the space given in Example 2. Finally, we do not know if there exists a Banach with numerical index 1 in which condition (c) is not satisfied.

**Problem 13 (**[12, Remark 3.5]**)** *Let* $X$ *be a Banach space with numerical index* $1$*. Does there exist a norming subset* $A$ *of* $S_{X^*}$ *such that* $|x^{**}(a^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every* $a^* \in A$*?*

Necessary conditions in the spirit of McGregor's result were given in 1999 by G. López, M. Martín, and R. Payá [53]. Their key idea was considering denting points instead of general extreme points. Recall that $x_0 \in B_X$ is said to be a *denting point* of $B_X$ if it belongs to slices of $B_X$ with arbitrarily small diameter. If $X$ is a dual space and the slices can be taken to be defined by weak*-continuous functionals, then we say that $x_0$ is a *weak*-denting point*.

**Proposition 7 (**[53, Lemma 1]**)** *Let $X$ be a Banach space with numerical index $1$. Then,*

*(a) $|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$ and every denting point $x \in B_X$.*

*(b) $|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every weak*-denting point $x^* \in B_{X^*}$.*

This result will play a key role in the next section.

Let us comment that, like McGregor original result, the conditions in Proposition 7 are not sufficient in the infinite-dimensional context. Indeed, the space $X = C([0,1], \ell_2)$ does not have numerical index 1, while $B_X$ has no denting points and there are no $w^*$-denting points in $B_{X^*}$. Actually, all the slices of $B_X$ and the $w^*$-slices of $B_{X^*}$ have diameter 2 (see [43, Lemma 2.2 and Example on p. 858], for instance).

Anyhow, if we have a Banach space $X$ such that $B_X$ has enough denting points (if $X$ has the Radon-Nikodým property, for instance), then item (a) in the above proposition combines with Proposition 6 to characterize the numerical index $1$ for $X$. The same is true for item (b) when $B_{X^*}$ has enough weak*-denting points (if $X$ is an Asplund space, for instance).

**Corollary 1 (**[57, Theorem 1] and [58, §1]**)** *Let $X$ be a Banach space.*

*(a) If $X$ has the Radon-Nikodým property, then the following are equivalent:*

- $(i)$ *$X$ has numerical index $1$.*
- $(ii)$ *$|x^*(x)| = 1$ for every $x^* \in \operatorname{ext}(B_{X^*})$ and every denting point $x$ of $B_X$.*
- $(iii)$ *$|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every $x^* \in \operatorname{ext}(B_{X^*})$.*

*(b) If $X$ is an Asplund space, then the following are equivalent:*

- $(i)$ *$X$ has numerical index $1$.*
- $(ii)$ *$|x^{**}(x^*)| = 1$ for every $x^{**} \in \operatorname{ext}(B_{X^{**}})$ and every weak*-denting point $x^* \in B_{X^*}$.*

In the remaining part of this section we discuss other kind of sufficient conditions for a Banach space to have numerical index 1.

The eldest of these properties was introduced in the fifties by O. Hanner [30]: a real Banach space has the *intersection property 3.2* (*3.2.I.P.* in short) if every collection of three mutually intersecting closed balls has nonempty intersection. The 3.2.I.P. was systematically studied by J. Lindenstrauss [50] and Å. Lima [48], and typical examples of spaces with this property are $L_1(\mu)$ and their isometric preduals. Real Banach spaces with the 3.2.I.P. have numerical index 1 since they fulfil Proposition 6.b (see [48, Corollary 3.3] and [50, Theorem 4.7]). The converse is false even in the finite-dimensional case (see [30, Remark 3.6] and [50, p. 47]).

Another isometric property, weaker than the 3.2.I.P. but still ensuring numerical index 1, was introduced by R. Fullerton in 1960 [24]. A real or complex Banach space is said to be a *CL-space* if its unit ball is the absolutely convex hull of every maximal convex subset of the unit sphere. If the unit ball is merely the closed absolutely convex hull of every maximal convex subset of the unit sphere, we say that the space is an *almost-CL-space* (J. Lindenstrauss [50] and Å. Lima [49]). Both definitions appeared only for real spaces, but they extend literally to the complex case. For general information, we refer to the already cited papers [48, 49, 50]; more recent results can be found in [63, 71]. Let us remark that the complex space $\ell_1$ is an almost-CL-space which is not a CL-space [63, Proposition 1], but we do not know if such an example exists in the real case.

**Problem 14** *Is there any real almost-CL-space which is not a CL-space?*

The fact that CL-spaces have numerical index 1 was observed by M. Acosta [1], and her proof extends easily to almost-CL-spaces (see [55, Proposition 12]). Actually, almost-CL-spaces fulfil condition (c) of Proposition 6 as shown in [63, Lemma 3]. In the converse direction, the basic examples of Banach spaces with numerical index 1 are known to be almost-CL-spaces (see [63] and [10, Theorem 32.9]). Moreover, all finite-dimensional spaces with numerical index 1 are CL-spaces [49, Corollary 3.7], and a Banach space with the Radon-Nikodým property and numerical index 1 is an almost-CL-space [57, Theorem 1]. Nevertheless, Banach spaces with numerical index 1 which are no almost-CL-spaces have been recently found. Actually, this happens with the space given in Example 2 [12, Example 3.4].

The last condition we would like to mention is a weakening of the concept of almost-CL-space introduced in [12]. A Banach space $X$ is said to be *lush* if for every $x, y \in S_X$ and every $\varepsilon > 0$, there exists $y^* \in S_{Y^*}$ such that

$$y \in S(B_X, y^*, \varepsilon) := \{z \in B_X \ : \ \operatorname{Re} y^*(z) > 1 - \varepsilon\}$$

and

$$\operatorname{dist}\big(x, \operatorname{co}\big(\mathbb{T}\, S(B_X, y^*, \varepsilon)\big)\big) < \varepsilon.$$

In the real case, the above definition is equivalent to the following one: for every $x, y \in S_X$ and every $\varepsilon > 0$, there exist $z \in S_X$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that

$$\|y + z\| > 2 - \varepsilon, \qquad |\gamma_1 - \gamma_2| = 2, \qquad \|x + \gamma_i\, z\| \leqslant 1 + \varepsilon \ \ (i = 1, 2).$$

See Figure 2 below for an interpretation of this property in dimension 2.

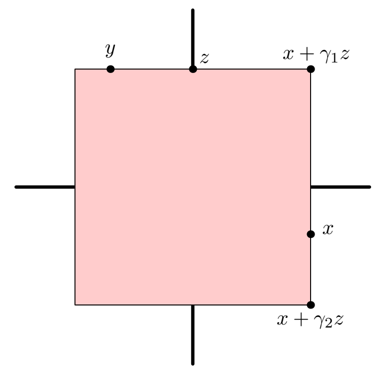


Figure 2. $\ell_\infty^2$ is lush

Almost-CL-spaces are clearly lush, and lush spaces have numerical index 1 [12, Proposition 2.2]. The converse of the first implication is not true, and once again the counterexample is the space $X$ given in Example 2: as we already mentioned, $X$ is not an almost-CL-space, but the original proof given in [12, Example 3.1] of the fact that $n(X) = 1$ passes through the lushness of the space. Actually, this example is only a specimen of a general family of lush subspaces of $C(K)$ introduced in [12], namely C-rich subspaces. A subspace $X$ of $C(K)$ is *C-rich* if for every nonempty open subset $U$ of $K$ and every $\varepsilon > 0$, there is a positive continuous norm-one function $h$ with support inside $U$, such that the distance from $h$ to $X$ is less than $\varepsilon$.

**Theorem 4 (**[12, Theorem 2.4]**)** *Let $K$ be a Hausdorff topological space and let $X$ be a C-rich subspace of $C(K)$. Then, $X$ is lush and, therefore, $n(X) = 1$.*

In particular, this occurs for finite codimensional subspaces of $C(K)$ when $K$ has no isolated points. Actually, the following result characterizes C-rich finite-codimensional subspaces of an arbitrary $C(K)$. We recall that the *support* of a regular measure $\mu \in C(K)^*$ is

$$\mathrm{supp}(\mu) = \bigcap \{C \subset K \ : \ C \text{ closed}, \ |\mu|(K \setminus C) = 0\} \, .$$

**Example 5 (**[12, Proposition 2.5]**)** *Let $K$ be a compact Hausdorff space and let $\mu_1, \ldots, \mu_n \in C(K)^*$. The subspace*

$$Y = \bigcap_{i=1}^{n} \ker \mu_i$$

*is C-rich if and only if $\bigcup_{i=1}^{n} \mathrm{supp}(\mu_i)$ does not intersect the set of isolated points of $K$.*

We do not know if the class of lush spaces exhausts the whole class of Banach spaces with numerical index 1.

**Problem 15** *Let $X$ be a Banach space with $n(X) = 1$. Is $X$ lush?*

# 4. Renorming and numerical index

In 2003, C. Finet, M. Martín, and R. Payá [23] studied the numerical index from the isomorphic point of view, i.e. they investigated the set $\mathcal{N}(X)$ of those values of the numerical index which can be obtained by equivalent renormings of a Banach space $X$. This study has a precedent in the 1974 paper [78] by K. Tillekeratne, where it is proved that every complex space of dimension greater than one can be renormed to achieve the minimum value of the numerical index; the same is true for real spaces.

**Proposition 8 (**[23, Proposition 1] and [78, Theorem 3.1]**)** *Let $X$ be a Banach space of dimension greater than one. Then $0 \in \mathcal{N}(X)$ in the real case, $\mathrm{e}^{-1} \in \mathcal{N}(X)$ in the complex case.*

One of the main aims of [23] is to show that $\mathcal{N}(X)$ is a interval for every Banach space $X$. To get this result, the authors use the continuity of the mapping carrying every equivalent norm on $X$ to its numerical index with respect to a metric taken from [10, §18].

**Proposition 9 (**[23, Proposition 2]**)** *$\mathcal{N}(X)$ is an interval for every Banach space $X$.*

As an immediate consequence of the above two results, we get the following.

**Corollary 2 (**[23, Corollary 3]**)** *If $1 \in \mathcal{N}(X)$ for a Banach space $X$ of dimension greater than one, then $\mathcal{N}(X) = [0, 1]$ in the real case and $\mathcal{N}(X) = [\mathrm{e}^{-1}, 1]$ in the complex case.*

Since $n(\ell_\infty^m) = 1$ for every $m$, the following particular case arises.

**Corollary 3 (**[78, Theorem 3.2]**)** *Let $m$ be an integer larger than 1. Then*

$$\mathcal{N}(\mathbb{R}^m) = [0, 1] \qquad \text{and} \qquad \mathcal{N}(\mathbb{C}^m) = [\mathrm{e}^{-1}, 1].$$

Now, one may ask if the above result is also true in the infinite-dimensional context, equivalently, whether or not every Banach space can be equivalently renormed to have numerical index 1. The answer is negative, as shown in the already cited paper [53].

**Theorem 5 (**[53, Theorem 3]**)** *Let $X$ be an infinite-dimensional real Banach space with $1 \in \mathcal{N}(X)$. If $X$ has the Radon-Nikodým property, then $X$ contains $\ell_1$. If $X$ is an Asplund space, then $X^*$ contains $\ell_1$.*

It follows that infinite-dimensional real reflexive spaces cannot be renormed to have numerical index 1. But even more is true.

**Corollary 4 (**[53, Corollary 5]**)** *Let $X$ be an infinite-dimensional real Banach space. If $X^{**}/X$ is separable, then $1 \notin \mathcal{N}(X)$.*

It is easy to explain how Theorem 5 was proved in [53]. Namely, the authors used Proposition 7, the well-known facts that the unit ball of a space with the Radon-Nikodým property has many denting points and that the dual unit ball of an Asplund space has many weak$^*$-denting points (see [11], for instance), and the following sufficient condition for a real Banach space to contain either $c_0$ or $\ell_1$.

**Lemma 1 (**[53, Proposition 2]**)** *Let $X$ be a real Banach space, and assume that there is an infinite set $A \subset S_X$ such that $|x^*(a)| = 1$ for every $a \in A$ and every $x^* \in \operatorname{ext}(B_{X^*})$. Then $X$ contains $c_0$ or $\ell_1$.*

Thus, the first open question in this line is the following.

**Problem 16** *Characterize those Banach spaces which can be equivalently renormed to have numerical index* 1.

We also propose to study separately necessary and sufficient conditions for a Banach space to be renormable with numerical index 1. With respect to necessary conditions, we have obtained two in the real case, namely Theorem 5 and Corollary 4. It is not known if they are valid in the complex case; actually, the following especial case remains open.

**Problem 17** *Does there exist an infinite-dimensional complex reflexive space which can be renormed to have numerical index* 1*?*

For more necessary conditions, we suggest to study the following two questions. The first one has a positive answer for real almost-CL-spaces [63, Theorem 5].

**Problem 18** *Let $X$ be an infinite-dimensional (real) Banach space satisfying that $1 \in \mathcal{N}(X)$. Does $X^*$ contain $\ell_1$?*

**Problem 19** *Let $X$ be an infinite-dimensional (real) Banach space satisfying that $1 \in \mathcal{N}(X)$. Does $X$ contain $c_0$ or $\ell_1$?*

Let us remark that we do not know of any non-trivial sufficient condition for a Banach space to be renormable with numerical index 1. We propose the following ones to be checked.

**Problem 20** *Let $X$ be a Banach space containing an infinite-dimensional subspace $Y$ with $1 \in \mathcal{N}(Y)$. Is it true that $1 \in \mathcal{N}(X)$?*

One may consider many especial cases.

**Problem 21** *Let $X$ be a Banach space containing a subspace isomorphic to either $c_0$, $\ell_1$, $C[0,1]$, or $L_1[0,1]$. Is it true that $1 \in \mathcal{N}(X)$?*

The following question is especially interesting since in view of Problem 18 it might lead to a characterization of Banach spaces that can be renormed to have numerical index 1.

**Problem 22** *Let $X$ be a Banach space such that $\ell_1 \subseteq X^*$. Is is true that $1 \in \mathcal{N}(X)$?*

We finish this section by showing that the value 1 of the numerical index is very particular. Indeed, it is proved in [23] that "most" Banach spaces can be renormed to achieve any possible value for the numerical index except eventually 1. Recall that a system $\{(x_\lambda, x_\lambda^*)\}_{\lambda\in\Lambda} \subset X \times X^*$ is said to be *biorthogonal* if $x_\lambda^*(x_\mu) = \delta_{\lambda,\mu}$ for $\lambda, \mu \in \Lambda$, and *long* if the cardinality of $\Lambda$ coincides with the density character of $X$.

**Theorem 6 (**[23, Theorem 10]**)** *Let $X$ be a Banach space admitting a long biorthogonal system. Then* $\sup \mathcal{N}(X) = 1$*. Therefore, when the dimension of $X$ is greater than one, $\mathcal{N}(X) \supset [0, 1[$ in the real case and $\mathcal{N}(X) \supset [e^{-1}, 1[$ in the complex case.*

Typical examples of Banach spaces admitting a long biorthogonal system are WCG spaces (see [18]). For instance, if $X^{**}/X$ is separable, then the Banach space $X$ is WCG (see [80, Theorem 3], for example) while, in the real case, $1 \notin \mathcal{N}(X)$ unless $X$ is finite-dimensional (see Corollary 4). Therefore, in many cases one of the inclusions of Theorem 6 becomes an equality.

**Corollary 5 (**[23, Corollary 11]**)** *Let $X$ be an infinite-dimensional real Banach space such that $X^{**}/X$ is separable. Then $\mathcal{N}(X) = [0, 1[$.*

Let us comment that Theorem 6 is proved by using a geometrical property that was introduced by J. Lindenstrauss in the study of norn-attaining operators [51] and called *property* $\alpha$ by W. Schachermayer [75]. It is known that, under the continuum hypothesis, there are Banach spaces which cannot be renormed with property $\alpha$ [27, 67]. Nevertheless, B. Godun and S. Troyanski proved in [27, Theorem 1] that this renorming is possible for Banach spaces admitting a long biorthogonal system; as far as we know, this is the largest class of spaces for which renorming with property $\alpha$ is possible.

The question arises if the assumption of having a long biorthogonal system in Theorem 6 can be dropped.

**Problem 23** *Is is true that* $\sup \mathcal{N}(X) = 1$ *for every Banach space $X$?*

It is also studied in [23] the relationship between the numerical index and the so-called *property* $\beta$ [51, 75]. Contrary to property $\alpha$, property $\beta$ is isomorphically trivial (J. Partington [69]), but it does not produce such a good result as Theorem 6. At least, it can be used to prove that $\mathcal{N}(X)$ does not reduces to a point when the dimension of $X$ is greater than one.

**Theorem 7 (**[23, Theorem 9]**)** *Let $X$ be a Banach space with* $\dim(X) > 1$*. Then $\mathcal{N}(X) \supset [0, 1/3[$ in the real case and $\mathcal{N}(X) \supset [\mathrm{e}^{-1}, 1/2[$ in the complex case.*

# 5. Real Banach spaces with numerical index zero

As we commented in the introduction, real Banach spaces underlying complex Banach spaces as well as real Hilbert spaces of dimension greater than one, have numerical index $0$. By Proposition 3, the absolute sum of such a space and any other real Banach space has numerical index $0$.

The general open question in this section is the following.

**Problem 24** *Find characterizations of Banach spaces with numerical index* $0$ *which do not involve operators.*

A sufficient condition which generalizes all the introductory examples is the following easy result of M. Martín, J. Merí and A. Rodríguez Palacios [60]. We say that a real vector space has a *complex structure* if it is the real space underlying a complex vector space.

**Proposition 10 (**[60, Proposition 2.1]**)** *Let $X$ be a real Banach space, and let $Y, Z$ be closed subspaces of $X$, with $Z \neq 0$. Suppose that $Z$ is endowed with a complex structure, that $X = Y \oplus Z$, and that the equality $\left\|y + \mathrm{e}^{i\rho} z\right\| = \|y + z\|$ holds for every $(\rho, y, z) \in \mathbb{R} \times Y \times Z$. Then $n(X) = 0$.*

It looks like if a necessary condition for numerical index $0$ could be the emergence of a subspace with some kind of complex structure. As a matter of fact, the following example shows that this is not the case.

**Example 6 (**[60, Example 2.2]**)** *There exists a real Banach space $X$ with numerical index $0$ which is polyhedral, i.e. the intersection of $B_X$ with any finite-dimensional subspace of $X$ is the convex hull of a finite set of points. Therefore, $X$ does not contain any isometric copy of $\mathbb{C}$.*

The above example has the additional interest that the numerical radius is a norm on $L(X)$, i.e. the only operator with numerical radius 0 is the zero operator. It could be the case that a Banach space in which there is a non-null operator with numerical radius 0 has a subspace with some kind of complex structure. We recall that a bounded linear operator $T$ on a (real or complex) Banach space $X$ is *skew-hermitian* if $\operatorname{Re} V(T) = \{0\}$; we write $\mathcal{Z}(X)$ for the (possibly null) closed subspace of $L(X)$ consisting of skew-hermitian operators on $X$. In the real case, $T$ is skew-hermitian if and only if $v(T) = 0$; when the space $X$ is complex, an operator $T$ is *hermitian* if $V(T) \subset \mathbb{R}$, i.e. the operator $i\,T$ is skew-hermitian. Hermitian operators have been deeply studied since the sixties and many results on Banach algebras depend on them; we refer to [9, 10] for more information. Also, skew-hermitian operators have been widely discussed in the seventies and eighties, especially in the finite-dimensional case; more information can be found in the papers by H. Rosenthal [73, 74] and references therein.

**Problem 25** *Let $X$ be a real space which has a non-null skew-hermitian operator. Does $X$ contain a subspace with a complex structure?*

Let us give a clarifying example. If $H$ is a $n$-dimensional Hilbert space, it is easy to check that $\mathcal{Z}(H)$ is the space of skew-symmetric operators on $H$ (i.e. $T^* = -T$ in the Hilbert space sense), so it identifies with the space of skew-symmetric matrixes $\mathcal{A}(n)$. It is a classical result from the theory of linear algebra that a $n \times n$ matrix $A$ belongs to $\mathcal{A}(n)$ if and only if $\exp(\rho A)$ is an orthogonal matrix for every $\rho \in \mathbb{R}$ (see [3, Corollary 8.5.10] for instance).

It is shown in [9, §3] that the above fact extends to general Banach spaces. Indeed, for an arbitrary Banach space $X$ and an operator $T \in L(X)$, by making use of the "exponential formula"

$$\sup \operatorname{Re} V(T) = \sup_{\alpha>0} \frac{\log \|\exp(\alpha T)\|}{\alpha},$$

it is easy to prove that the following are equivalent:

$(i)$ $T$ is skew-hermitian,

$(ii)$ $\exp(\rho T)$ is an onto isometry for every $\rho \in \mathbb{R}$.

From now on, we will restrict ourselves to finite-dimensional spaces. Here, the above result reminds the theory of Lie groups and Lie algebras, for which the group of orthogonal matrices and the space of skew-symmetric matrices are distinguished examples. Actually, the group of all the isometries on a finite-dimensional real space $X$ is a Lie group whose associated Lie algebra is $\mathcal{Z}(X)$ [73, Theorem 1.4 and Proposition 1.5]. With this in mind and using results from the theory of Lie groups, it is proved in [73, Theorem 3.8] that the following are equivalent for a finite-dimensional real space $X$:

$(i)$ the numerical index of $X$ is 0,

$(ii)$ there are infinitely many isometries on $X$.

The main open question in this section is the following.

**Problem 26** *Describe the finite-dimensional real Banach spaces with numerical index $0$.*

The next result follows this line. In particular, it shows that finite-dimensional normed spaces with numerical index 0 wear some kind of complex structure.

**Theorem 8 (**[73, Corollary 3.7] and [60, Theorem 2.4]**)** *Let $X$ be a finite-dimensional real Banach space. Then, the following are equivalent:*

$(i)$ *The numerical index of $X$ is zero.*

$(ii)$ *There are nonzero complex vector spaces $X_1, \dots, X_m$, a real vector space $X_0$, and positive integer numbers $q_1, \dots, q_m$ such that $X = X_0 \oplus X_1 \oplus \cdots \oplus X_m$ and*

$$\left\| x_0 + \mathrm{e}^{iq_1\rho} x_1 + \cdots + \mathrm{e}^{iq_m\rho} x_m \right\| = \|x_0 + x_1 + \cdots + x_m\|$$

*for all $\rho \in \mathbb{R}$, $x_j \in X_j$ $(j = 0, 1, \dots, m)$.*

Some remarks are pertinent. First, the above result shows that a finite-dimensional space with numerical index 0 contains complex subspaces (at least one), whose complex structures are well related one to the others and to the non-complex part. Second, the above result is taken literally from the paper [60], but the same equivalence with arbitrary real numbers $q_1, \dots, q_m$ had appeared in the 1985 Rosenthal's paper [73]; the fact that the $q_j$'s can be taken integers is new from [60] and it uses the classical Kronecker's Approximation Theorem (see [32, Theorem 442] for instance). Third, it can be asked if the number of complex spaces in the theorem can be reduced to one, and so Proposition 10 would be an equivalence in the finite-dimensional case. The answer is negative.

**Example 7 (**[60, Example 2.8]**)** *The space $\mathbb{R}^4$ with norm*

$$\|(a,b,c,d)\| = \frac{1}{4} \int_0^{2\pi} \left| \mathrm{Re}\left( \mathrm{e}^{2it}(a+ib) + \mathrm{e}^{it}(c+id) \right) \right| \, dt \qquad (a,b,c,d \in \mathbb{R})$$

*has numerical index* 0*, but the number of complex spaces in Theorem 8.ii cannot be reduced to one.*

Such an example as the above is not possible in dimensions two or three. Actually, in this case, Theorem 8 takes a more suitable form [73, Theorem 3.1]. Let $X$ be a real Banach space with numerical index 0.

(a) If $\dim(X) = 2$, then $X$ is isometrically isomorphic to the two-dimensional real Hilbert space.

(b) If $\dim(X) = 3$, then $X$ is an absolute sum of $\mathbb{R}$ and the two-dimensional real Hilbert space.

Our next aim is to discuss some questions related to the Lie algebra of skew-hermitian operators $\mathcal{Z}(X)$ of an arbitrary $n$-dimensional space which. The main related open question is the following.

**Problem 27** *Figure out what are the possible values for the dimension of $\mathcal{Z}(X)$ when $\dim(X) = n$.*

Let us fix a $n$-dimensional real Banach space $X$. It follows from a theorem of Auerbach [72, Theorem 9.5.1], that there exists an inner product $(\cdot|\cdot)$ on $X$ such that every skew-hermitian operator on $X$ remains skew-hermitian (hence skew-symmetric) on $H := (X, (\cdot|\cdot))$. Then, by just fixing an orthonormal basis of $H$, we get an identification of $\mathcal{Z}(X)$ with a Lie subalgebra of the Lie algebra $\mathcal{A}(n)$. Therefore,

$$\dim\left(\mathcal{Z}(X)\right) \leqslant \frac{n(n-1)}{2}.$$

The equality holds if and only if $X$ is a Hilbert space (see [73, Theorem 3.2] or [60, Corollary 2.7]). It is a good question whether or not all the intermediate numbers are possible values for the dimension of $\mathcal{Z}(X)$. The answer is negative, as a consequence of Theorem 3.2 in Rosenthal's paper [73], which reads as follows.

(a) If $\dim\left(Z(X)\right) > \frac{(n-1)(n-2)}{2}$, then $X$ is a Hilbert space and, therefore, $\dim\left(Z(X)\right) = \frac{n(n-1)}{2}$.

(b) $\dim\left(Z(X)\right) = \frac{(n-1)(n-2)}{2}$ if and only if $X$ is a non-Euclidean absolute sum of $\mathbb{R}$ and a Hilbert space of dimension $n-1$.

For low dimensions, Problem 27 has been solved in [74]. When the dimension of $X$ is 3, the above result leaves only the following possible values for the dimension of $\mathcal{Z}(X)$: 0 as for $X = \ell_\infty^3$, 1 as for $\mathbb{R} \oplus_1 \mathbb{C}$, and 3 as for $\ell_2^3$. When the dimension of $X$ is 4, the possible values of the dimension of $\mathcal{Z}(X)$ allowed by the above result are $0, 1, 2, 3, 6$; all of them are possible [74, pp. 443]. The first dimension in which Problem 27 is open is $n = 5$.

**Problem 28** *What are the possible values for the dimension of $\mathcal{Z}(X)$ when $X$ is a 5-dimensional real Banach space?*

## 6. Asymptotic behavior of the set of finite-dimensional spaces with numerical index one

Let us start the section by recalling that most of the sufficient and the necessary conditions for having numerical index 1 given in section 3 are actually characterizations in the finite-dimensional case. For the convenience of the reader, we summarize them in the following paragraph.

*Let $X$ be a finite-dimensional real normed space. Then, the following are equivalent.*

$(i)$ *$X$ has numerical index* $1$.

$(ii)$ $|x^*(x)| = 1$ *for all the extreme points $x^*$ of $B_{X^*}$ and $x$ of $B_X$.*

$(iii)$ *$X$ is a CL-space, i.e. $B_X = \operatorname{co}(F \cup -F)$ for every maximal convex subset $F$ of $S_X$.*

$(iv)$ *$X$ is lush, i.e. for every $x, y \in S_X$, there are $z \in S_X$ and $\gamma_1, \gamma_2 \in \mathbb{R}$ such that $\|y + z\| = 2$, $|\gamma_1 - \gamma_2| = 2$, and $\|x + \gamma_i\, z\| \leqslant 1$ $(i = 1, 2)$.*

Our aim in this section is to consider the asymptotic behavior (as the dimension grows to infinity) of some parameters related to the Banach-Mazur distance for the family of finite-dimensional real normed spaces with numerical index 1. Let us write $\mathcal{N}_m$ for the space of all $m$-dimensional normed spaces endowed with the *Banach-Mazur distance*

$$d(X,Y) = \inf\{\|T\|\,\|T^{-1}\| \ : \ T : X \longrightarrow Y \text{ isomorphism}\} \qquad (X, Y \in \mathcal{N}_m),$$

and let us write $\mathcal{M}_m$ for the subset consisting of those $m$-dimensional spaces with numerical index 1. Our aim is to study some questions related to these two spaces. As far as we know, the first result of this kind was given very recently by T. Oikhberg [68].

**Theorem 9 (**[68, Theorem 4.1]**)** *There exists a universal positive constant $c$ such that*

$$d(X, \ell_2^m) \geqslant c\, m^{\frac{1}{4}}$$

*for every $m \geqslant 1$ and every $X \in \mathcal{M}_m$.*

It is well-known that $d(\ell_1^m, \ell_2^m) = d(\ell_\infty^m, \ell_2^m) = \sqrt{m}$ for every $m > 1$ (see [25, pp. 720] for instance). Therefore, the following question arises naturally.

**Problem 29** *Does there exists a universal constant $c > 0$ such that*

$$d(X, \ell_2^m) \geqslant c\,\sqrt{m}$$

*for every $m \geqslant 1$ and every $X \in \mathcal{M}_m$?*

It was observed in [68, pp. 622] that the answer to this question is positive for the spaces with the 3.2.I.P., even with $c = 1$. This is a consequence of the 1981 result by A. Hansen and Å. Lima [31] that these spaces are constructed starting from the real line and producing successively $\ell_\infty$ and/or $\ell_1$ sums. But, as we already mentioned, not every element of $\mathcal{M}_m$ has the 3.2.I.P.

Finally, we would like to propose some related questions.

**Problem 30** *What is the diameter of $\mathcal{M}_m$? Is it (asymptotically) close to the diameter of $\mathcal{N}_m$?*

**Problem 31** *What is the biggest possible distance from an element of $\mathcal{N}_m$ to the set $\mathcal{M}_m$?*

## 7. Relationship to the Daugavet property.

In every Banach space with the Radon-Nikodým property (in particular in every reflexive space) the unit ball must have denting points. There are Banach spaces $X$ (as $C[0,1]$, $L_1[0,1]$, and many others) with an extremely opposite property: for every $x \in S_X$ and for arbitrarily small $\varepsilon > 0$, the closure of

$$\mathrm{co}\big(B_X \setminus (x + (2-\varepsilon)B_X)\big)$$

equals to the whole $B_X$ (see Figure 3 below). This geometric property of the space is equivalent to the following exotic property of operators on $X$: for every compact operator $T : X \longrightarrow X$, the so-called *Daugavet equation*

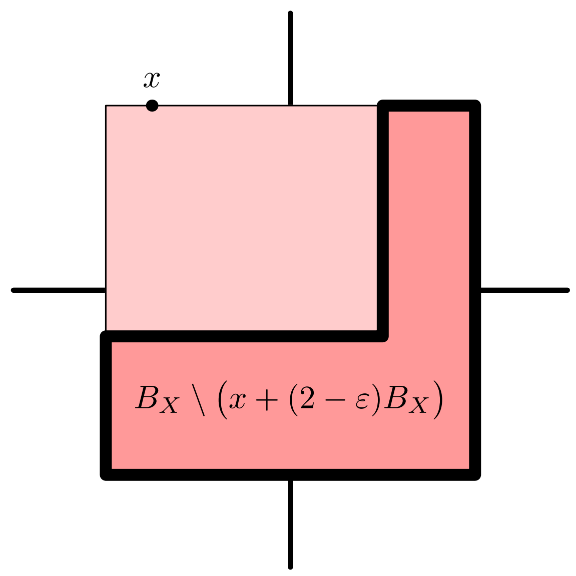


Figure 3. $\ell_\infty^2$ does not have the Daugavet property

$$\|\mathrm{Id} + T\| = 1 + \|T\| \tag{DE}$$

holds. This property of $C[0,1]$ was discovered by I. Daugavet in 1963 and is called the *Daugavet property* [42, 43]. Over the years, the validity of the Daugavet equation was proved for some classes of operators on various spaces, including weakly compact operators on $C(K)$ and $L_1(\mu)$ provided that $K$ is perfect and $\mu$ does not have any atoms (see [81] for an elementary approach), and on certain function algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions $H^\infty$ [82, 84]. In the nineties, new ideas were infused into this field and the geometry of Banach spaces having the Daugavet property was studied; we cite the papers of V. Kadets, R. Shvidkoy, G. Sirotkin, and D. Werner [43] and R. Shvidkoy [77] as representatives. Let us comment that the original definition of Daugavet property given in [42, 43] only required rank-one operators to satisfy (DE) and, in such a case, this equation also holds for every bounded operator which does not fix a copy of $\ell_1$ [77].

Although the Daugavet property is of isometric nature, it induces various isomorphic restrictions. For instance, a Banach space with the Daugavet property contains $\ell_1$ [43], it does not have unconditional basis (V. Kadets [38]) and, moreover, it does not isomorphically embed into an unconditional sum of Banach spaces without a copy of $\ell_1$ [77]. It is worthwhile to remark that the latest result continues a line of generalization ([39], [41], [43]) of the well known theorem by A. Pełczyński [70] that $L_1[0,1]$ (and so $C[0,1]$) does not embed into a space with unconditional basis.

The state-of-the-art on the Daugavet property can be found in [83]; for very recent results we refer the reader to [5, 6, 37, 40, 44] and references therein.

Let us explain the relation between (DE) and the numerical range of an operator. In the aforementioned paper [20] by J. Duncan, C. McGregor, J. Pryce, and A. White, it was deduced from formula (3) on page 160 that an operator $T$ on a Banach space $X$ satisfies (DE) if and only if $\sup \operatorname{Re} V(T) = \|T\|$. Therefore, $v(T) = \|T\|$ if and only the following equality holds

$$\max_{\omega\in\mathbb{T}} \|\mathrm{Id} + \omega\, T\| = 1 + \|T\| \tag{aDE}$$

(see [61, Lemma 2.3] for an explicit proof). Therefore, it was known since 1970 that every bounded linear operator on $C(K)$ or $L_1(\mu)$ satisfies (aDE), a fact that was rediscovered and reproved in some papers from the eighties and nineties as the ones by Y. Abramovich [2], J. Holub [34], and K. Schmidt [76].

This latest equation was named as the *alternative Daugavet equation* by M. Martín and T. Oikhberg in [61], where the following property was introduced. A Banach space $X$ is said to have the *alternative Daugavet property* if every rank-one operator on $X$ satisfies (aDE). In such a case, every weakly compact operator on $X$ also satisfies (aDE) [61, Theorem 2.2]. Therefore, $X$ has the alternative Daugavet property if and only if $v(T) = \|T\|$ for every weakly compact operator $T$ on $X$.

Let us comment that, contrary to the Daugavet property, this property depends upon the base field (e.g. $\mathbb{C}$ has it as a complex space but not as a real space). For more information on the alternative Daugavet property we refer to the already cited paper [61] and also to [58]. From the former one we take the following geometric characterizations of the alternative Daugavet property.

**Proposition 11 (**[61, Propositions 2.1 and 2.6]**)** *Let $X$ be a Banach space. Then, the following are equivalent.*

$(i)$ *$X$ has the alternative Daugavet property.*

$(ii)$ *For all $x_0 \in S_X$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there is some $x \in S_X$ such that*

$$|x_0^*(x)| \geqslant 1 - \varepsilon \qquad \text{and} \qquad \|x + x_0\| \geqslant 2 - \varepsilon.$$

$(ii^*)$ *For all $x_0 \in S_X$, $x_0^* \in S_{X^*}$ and $\varepsilon > 0$, there is some $x^* \in S_{X^*}$ such that*

$$|x^*(x_0)| \geqslant 1 - \varepsilon \qquad \text{and} \qquad \|x^* + x_0^*\| \geqslant 2 - \varepsilon.$$

$(iii)$ $B_X = \overline{\mathrm{co}}\Big(\mathbb{T}\left[B_X \setminus \big(x + (2-\varepsilon)B_X\big)\right]\Big)$ *for every $x \in S_X$ and every $\varepsilon > 0$ (see Figure 4 below).*

$(iii^*)$ $B_{X^*} = \overline{\mathrm{co}}^{w^*}\Big(\mathbb{T}\left[B_{X^*} \setminus \big(x^* + (2-\varepsilon)B_{X^*}\big)\right]\Big)$ *for every $x^* \in S_{X^*}$ and every $\varepsilon > 0$.*

*(iv)* $B_{X^*\oplus_\infty X^{**}} = \overline{\mathrm{co}}^{w^*}\ \big(\{(x^*, x^{**}) \,:\, x^* \in \mathrm{ext}(B_{X^*}),\ x^{**} \in \mathrm{ext}(B_{X^{**}}),\ |x^{**}(x^*)| = 1\}\big)$.

It is clear that both spaces with the Daugavet property and spaces with numerical index $1$ have the alternative Daugavet property. Both converses are false: the space $c_0 \oplus_1 C([0,1], \ell_2)$ has the alternative Daugavet property but fails the Daugavet property and its numerical index is not 1 [61, Example 3.2]. Nevertheless, under certain isomorphic conditions, the alternative Daugavet property forces the numerical index to be $1$.

**Proposition 12 (**[53, Remark 6]**)** *Let $X$ be a Banach space with the alternative Daugavet property. If $X$ has the Radon-Nikodým property or $X$ is an Asplund space, then $n(X) = 1$.*

With this result in mind, one realizes that the necessary conditions for a real Banach space to be renormed with numerical index 1 given in section $4$ (namely Theorem 5 and Corollary 4), can be written in terms of the alternative Daugavet property. Even more, in the proof of Proposition 7 given in [53], only rank-one operators are used and, therefore, it can be also written in terms of the alternative Daugavet property.

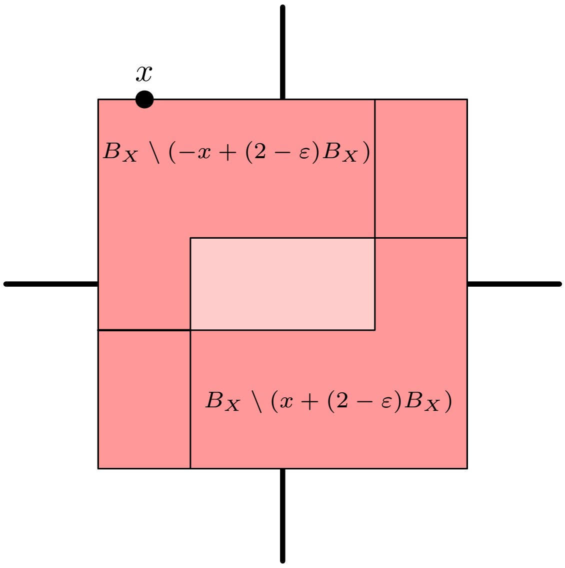


Figure 4. $\ell_\infty^2$ has the alternative Daugavet property

**Proposition 13 (**[53, Lemma 1 and Remark 6]**)** *Let $X$ be a Banach space with the alternative Daugavet property. Then,*

*(a)* $|x^{**}(x^*)| = 1$ *for every* $x^{**} \in \operatorname{ext}(B_{X^{**}})$ *and every weak*$^*$*-denting point* $x^* \in B_{X^*}$.

*(b)* $|x^*(x)| = 1$ *for every* $x^* \in \operatorname{ext}(B_{X^*})$ *and every denting point* $x \in B_X$.

**Proposition 14 (**[61, Remark 2.8]**)** *Let $X$ be an infinite-dimensional real Banach space with the alternative Daugavet property. If $X$ has the Radon-Nikodým property, then $X$ contains $\ell_1$. If $X$ is an Asplund space, then $X^*$ contains $\ell_1$. In particular, $X^{**}/X$ is not separable.*

The above two results give us an indication of why it is difficult to find characterizations of Banach spaces with numerical index $1$ that do no involve operators. Indeed, it is not easy to construct noncompact operators on an abstract Banach space. Thus, when one uses the assumption that a Banach space has numerical index $1$, only the alternative Daugavet property can be easily exploited. Of course, things are easier if one is working in a context where the alternative Daugavet property ensures numerical index $1$, as it happens with Asplund spaces and spaces with the Radon-Nikodým property. Therefore, it would be desirable to find more isomorphic properties ensuring that the alternative Daugavet property implies numerical index $1$. One possibility is the following.

**Problem 32** *Let $X$ be a Banach space which does not contain any copy of $\ell_1$ and having the alternative Daugavet property. Is it true that $n(X) = 1$?*

An affirmative answer to the above question might come from the following more ambitious one.

**Problem 33** *Let $X$ be a Banach space with the alternative Daugavet property. Is it true that $v(T) = \|T\|$ for every operator $T \in L(X)$ fixing no copy of $\ell_1$?*

On the other hand, it is not possible to find isomorphic properties ensuring that the alternative Daugavet property and the Daugavet property are equivalent.

**Proposition 15 (**[61, Corollary 3.3]**)** *Let $X$ be a Banach space with the alternative Daugavet property. Then there exists a Banach space $Y$, isomorphic to $X$, which has the alternative Daugavet property but fails the Daugavet property.*

We may then look for isometric conditions that allow passing from the alternative Daugavet property to the Daugavet property. Having a complex structure could be such a condition.

**Problem 34** *Let $X$ be a complex Banach space such that $X_{\mathbb{R}}$ has the alternative Daugavet property. Does it follow that $X$ (equivalently $X_{\mathbb{R}}$) has the Daugavet property?*

# 8. The polynomial numerical indices

In 1968, the concept of numerical range of operators was extended to arbitrary continuous functions from the unit sphere of a real or complex Banach space into the space by F. Bonsall, B. Cain, and H. Schneider [8] in the obvious way. They showed that the numerical range of a bounded linear operator is always connected, and the same is true for arbitrary continuous functions with only one trivial exception: when the space has dimension one over $\mathbb{R}$. Three years later, L. Harris [33] studied various possible numerical ranges for holomorphic functions on complex Banach spaces and got some deep results on them.

In this section, we concentrate on homogeneous polynomials on real or complex spaces. Let us give the necessary definitions; we refer the reader to the book [19] by S. Dineen for background on polynomials. Given a Banach space $X$ and a positive integer $k$, a mapping $P : X \longrightarrow X$ is called a (continuous) *$k$-homogeneous polynomial* on $X$ if there is an $k$-linear continuous mapping $A : X \times \cdots \times X \longrightarrow X$ such that $P(x) = A(x, \ldots, x)$ for every $x \in X$. A *polynomial* on $X$ is a finite sum of homogeneous polynomials. The space $\mathcal{P}(X; X)$ of all polynomials on $X$ is normed by

$$\|P\| = \sup_{x \in B_X} \|P(x)\| \qquad \big(P \in \mathcal{P}(X; X)\big).$$

We write $\mathcal{P}(^k X; X)$ to denote the subspace of $\mathcal{P}(X; X)$ of those $k$-homogeneous polynomials on $X$, which is a Banach space. The numerical range of $P \in \mathcal{P}(X; X)$ is the set of scalars

$$V(P) := \{x^*(P(x)) \ : \ x \in S_X, \ x^* \in S_{X^*}, \ x^*(x) = 1\},$$

and the numerical radius of $P$ is

$$v(P) := \sup\{|\lambda| \ : \ \lambda \in V(P)\}.$$

As in the linear case, it is natural to define the *polynomial numerical index of order $k$* of $X$ to be the constant

$$\begin{aligned} n^{(k)}(X) &:= \inf\big\{v(P) \ : \ P \in \mathcal{P}(^k X; X), \|P\| = 1\big\} \\ &= \max\big\{M \geqslant 0 \ : \ \|P\| \leqslant M v(P) \ \forall P \in \mathcal{P}(^k X; X)\big\}\,; \end{aligned}$$

of course, $n^{(1)}(X)$ coincides with the usual numerical index of the space $X$. This definition was introduced very recently by Y. Choi, D. García, S. Kim, and M. Maestre [13]. Note that $0 \leqslant n^{(k)}(X) \leqslant 1$, and that $n^{(k)}(X) > 0$ if and only if $v(\,\cdot\,)$ is a norm on $\mathcal{P}(^k X; X)$ equivalent to the usual norm.

The first result that we would like to mention here is an extension of Glickfeld's result for linear operators [26] given by L. Harris [33, Theorem 1]: for complex spaces, the numerical radius is always an equivalent norm in the space of $k$-homogeneous polynomials. More concretely, if $X$ is a complex Banach space and $k \geqslant 2$, then

$$n^{(k)}(X) \geqslant \exp\left(\frac{k\,\log(k)}{1-k}\right). \tag{9}$$

It was also proved in [33, §7] that the above inequalities are sharp. In the real case, Harris' result above is false since the polynomial numerical indices of a real space may vanish (see Example 8.b below).

The next result shows the behavior of the polynomial numerical index when the degree grows.

**Proposition 16 (**[13, Proposition 2.5]**)** *Let $X$ be a Banach space and let $k$ be a positive integer. Then*

$$n^{(k+1)}(X) \leqslant n^{(k)}(X).$$

Next we list several examples of Banach spaces for which there is some information on their polynomial numerical indices.

**Examples 8 (**[13], [14] and [15]**)**

*(a)* $n^{(k)}(\mathbb{K}) = 1$ *for every* $k \in \mathbb{N}$.

*(b) If $H$ is a real Hilbert space of dimension greater than* 1*, then* $n^{(k)}(H) = 0$ *for every* $k \in \mathbb{N}$.

*(c) If $H$ is a complex Hilbert space of dimension greater than* 1*, then* $1/4 \leqslant n^{(2)}(H) \leqslant 1/2$ *for every* $k \in \mathbb{N}$.

*(d) In the complex case, all the polynomial numerical indices of an $L_1$-predual are equal to* 1*. In particular, this is the case for the complex spaces $c_0$, $\ell_\infty$ and, more generally, $C(K)$.*

*(e)* $n^{(2)}(\ell_1) \leqslant 1/2$ *in the real as well as in the complex case.*

*(f) For every $k \geqslant 2$, the real spaces $c_0$, $\ell_\infty$, $c$, and $\ell_\infty^m$ with $m \geqslant 2$, have numerical index of order $k$ smaller than* 1.

*(g) Actually, if $K$ is a non-perfect compact Hausdorff space with at least two points, then the real space $C(K)$ has numerical index of order $k$ smaller than* 1 *for every* $k \geqslant 2$.

The above examples show that the polynomial numerical indices distinguish between $L$-spaces and $M$-spaces in the complex case, and this is not possible if we only use the usual (linear) numerical index. They also show that, unlike the linear case, there is no relationship between the polynomial numerical indices of a Banach space and the ones of its dual (see items (d) and (e) above). Nevertheless, there is a relationship with the polynomial numerical indices of the bidual.

**Proposition 17 (**[13, Corollary 2.15]**)** *Let $X$ be a Banach space. Then, we have*

$$n^{(k)}(X^{**}) \leqslant n^{(k)}(X)$$

*for every* $k \geqslant 2$.

We do not know if the above inequalities are actually equalities.

**Problem 35** *Is there a Banach space $X$ such that* $n^{(2)}(X^{**}) < n^{(2)}(X)$*?*

Let us present now more open questions arising from Examples 8.

**Problem 36** *Compute the polynomial numerical indices of complex Hilbert spaces and of* $\ell_1$.

**Problem 37** *Is* $n^{(2)}\big(C[0,1]\big) = 1$ *in the real case? Is* $n^{(2)}\big(L_1[0,1]\big) = 1$ *in the real and/or in the complex case?*

The behavior of polynomial numerical indices under direct sums and the indices of vector-valued continuous functions spaces were also studied in [13]. The results can be summarizedas follows.

**Proposition 18 (**[13, Propositions 2.8 and 2.10]**)** *Let $k$ be a positive integer.*

*(a) If $\{X_\lambda :\ \lambda \in \Lambda\}$ is any family of Banach spaces, then*

$$n^{(k)}\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{c_0}\Big) \leqslant \inf_\lambda n^{(k)}(X_\lambda)$$
$$n^{(k)}\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_1}\Big) \leqslant \inf_\lambda n^{(k)}(X_\lambda)$$
$$n^{(k)}\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}\Big) \leqslant \inf_\lambda n^{(k)}(X_\lambda).$$

*(b) If $X$ is a Banach space and $K$ is a compact Hausdorff space, then*

$$n^{(k)}\big(C(K,X)\big) \leqslant n^{(k)}(X).$$

Examples 8 show that the inequalities in item (a) above are not always equalities in the real case; just take $\Lambda = \mathbb{N}$ and $X_n = \mathbb{R}$ for every $n \in \mathbb{N}$. In the complex case, the same is true for the inequality involving $\ell_1$-sums ($\Lambda = \mathbb{N}$ and $X_n = \mathbb{C}$ for every $n \in \mathbb{N}$), but we do not know the answer for the other sums.

**Problem 38** *Let $\{X_\lambda :\ \lambda \in \Lambda\}$ be a family of complex Banach spaces. Is is true that*

$$n^{(k)}\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{c_0}\Big) = n^{(k)}\Big(\big[\oplus_{\lambda\in\Lambda} X_\lambda\big]_{\ell_\infty}\Big) = \inf_\lambda n^{(k)}(X_\lambda)\ ?$$

For item (b) of Proposition 18 the situation is similar. For instance, the real space $C(\{1,2\},\mathbb{R})$ does not have the same polynomial numerical indices as $\mathbb{R}$. We do not know if there exists an example of the same kind in the complex case.

**Problem 39** *Let $X$ be a complex Banach space. Is $n^{(k)}\big(C(K,X)\big)$ equal to $n^{(k)}(X)$?*

For spaces of Bochner-measurable integrable or essentially bounded vector valued functions no results are yet available.

**Problem 40** *Let $X$ be a Banach space, $\mu$ a positive $\sigma$-finite measure and $k \geqslant 2$. Is there any relationship between $n^{(k)}\big(L_1(\mu,X)\big)$ (resp. $n^{(k)}\big(L_\infty(\mu,X)\big)$) and $n^{(k)}(X)$?*

Item (a) in Proposition 18 can be used to produce the following nice example.

**Example 9** *There exists a complex Banach space $X$ such that*

$$n^{(k)}(X) = \exp\left(\frac{k\ \log(k)}{1-k}\right)$$

*for every $k \geqslant 2$, i.e. all the inequalities in* (9) *are simultaneously equalities.*

PROOF. For each positive integer $k \geqslant 2$, let $X_k$ be the two-dimensional complex Banach space given in [33, §7] which satisfies the required equality for this $k$. Then, the space

$$X = \big[\oplus_{k\geqslant 2} X_k\big]_{c_0}$$

satisfies

$$n^{(k)}(X) \leqslant n^{(k)}(X_k) = \exp\left(\frac{k\ \log(k)}{1-k}\right)$$

by Proposition 18.a, and the other inequality always holds. ■

After presenting the known results and open questions related to the computing of the polynomial numerical indices, we would like to discuss the isometric or structural consequences that these indices may have on a Banach space. Actually, we do not know of any result in this line, so we simply state conjectures and open questions.

First, we do not know what are precisely the values that the polynomial numerical index may take.

**Problem 41** *For a given $k \geqslant 2$, describe the sets*

$$\Big\{ n^{(k)}(X) \ :\ X \text{ real Banach space} \Big\} \qquad \text{and} \qquad \Big\{ n^{(k)}(X) \ :\ X \text{ complex Banach space} \Big\}.$$

We may also ask for the class of Banach spaces for which the the $k$-order numerical index is one of the extreme values.

**Problem 42** *Study the real Banach spaces $X$ satisfying $n^{(2)}(X) = 0$. For instance, do they satisfy that $n(X) = 0$? What is the answer for finite-dimensional spaces?*

**Problem 43** *Characterize the complex Banach spaces $X$ satisfying $n^{(k)}(X) = 1$ for all $k \geqslant 2$. In particular, do they always contain a copy of $c_0$ in the infinite-dimensional case?*

**Problem 44** *Is there any real Banach space $X$ different from $\mathbb{R}$ such that $n^{(2)}(X) = 1$?*

It would be desirable to get more information on the non-increasing sequence $\{n^{(k)}(X)\}_{k\in\mathbb{N}}$ of all polynomial numerical indices of a Banach space $X$. For instance, in view of the Examples 8, the following question arises.

**Problem 45** *Is there any Banach space $X$ for which $\lim\limits_{k\to\infty} n^{(k)}(X) \neq 0, 1$?*

To finish the section, let us mention that Y. Choi, D. García, M. Martín, and M. Maestre have extended the study of the Daugavet equation to polynomials on a Banach space in the very recent paper [14]. Given a Banach space $X$, a polynomial $P \in \mathcal{P}(X;X)$ satisfies the Daugavet equation if

$$\|\mathrm{Id} + P\| = 1 + \|P\|,$$

and it satisfies the alternative Daugavet equation if

$$\max_{\omega\in\mathbb{T}} \|\mathrm{Id} + \omega\, P\| = 1 + \|P\|.$$

Daugavet and alternative Daugavet properties are also translated in [14] to the polynomial setting by direct generalization of the linear case. A Banach space $X$ has the *$k$-order Daugavet property* (resp. the *$k$-order alternative Daugavet property*) if every rank-one $k$-homogeneous polynomial on $X$ satisfies the Daugavet equation (resp. the alternative Daugavet equation). These properties are related to the polynomial numerical indices since, as in the linear case, for a polynomial $P$ we have that

(a) $P$ satisfies the Daugavet equation if and only if $\|P\| = \sup \mathrm{Re}\, V(P)$,

(b) $P$ satisfies the alternative Daugavet equation if and only if $\|P\| = v(P)$

(see [14, Proposition 1.3]). Let us remark that part of the information given in Examples 8 comes from results on the Daugavet and the alternative Daugavet properties [14].

We conclude this paper by mentioning the following rather surprising result.

**Proposition 19 (**[14, Proposition 3.3 and Remark 3.6]**)** *Let $X$ be a Banach space and let $k \geqslant 2$. Then, the Daugavet equation and the alternative Daugavet equations are equivalent in $\mathcal{P}(^k X; X)$ in the following two cases:*

*(a) if the base field is $\mathbb{C}$, or*

*(b) if the base field is $\mathbb{R}$ and $k$ is even.*

*If $k$ is odd, then the Daugavet and the alternative Daugavet equation are not equivalent in $\mathcal{P}(^k\mathbb{R};\mathbb{R})$.*

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