A NEW LOOK AT SOME CLASSIC PROBLEMS IN ORTHOGONAL POLYNOMIALS

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This paper is dedicated to the founders of the theory and applications of orthogonal polynomials.

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Until recently there were two kinds of orthogonality commonly accepted: that generated by a real weight function and that generated by a complex weight function with integration taken along a path in the complex plane. The authors following R. D. Morton and A. M. Krall [10] research work are discussing the orthogonality from the point of view of distribution theory. In doing so a third and new kind of orthogonality is pointed out.

Las investigaciones clásicas sobre ortogonalidad de polinomios eran de dos géneros, según que la función de peso fuese real o compleja, y en este caso la integración era a lo largo de una trayectoria en el plano complejo. Siguiendo a Morton y Krall un tercer género de ortogonalidad está siendo investigado actualmente, bajo el punto de vista de la teoría de las distribuciones. En este nuevo marco se encaja la actual memoria.

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1. Introduction

Notwithstanding the fact that the number of papers devoted to *Special functions* (33XX of the AMS subject classification scheme 1979) is sensibly diminishing in our age, nowadays continue to appear in the same parallel direction not only basic books devoted to studies of orthogonal polynomials, as for instance the excellent and quite recent books: «Padé-Type Approximation and general Orthogonal Polynomials» [17] and «Orthogonal Polynomials» [21], where many results appear for the first time or are applied within a great deal of technical development to the general classes of orthonormal polynomials, respectively, but also quite a large set of papers concerning different special cases of orthogonal polynomials.

Before going to develop our enough comprehensive «New Look at some Classic Problems in Orthogonal Polynomials» let us quote below some of the recent papers published in 1979 and 1980 elaborated in the domain of *Special functions* by other researchers:

1.° «Some properties of q-Hermite polynomials» and «A proof of Mehler's formula for q-Hermite polynomials» [22], which are intimately connected with theta functions; 2.° «Zur Charakterisierung von Orthogonal Polynomen in zwei Veränderlichen» [23]; 3.° «A proof of Dunkel's addition theorem for Krawtchouk polynomials» [24], which are orthogonal with respect to the binomial distribution; 4.° «Laguerre (ortogonal) polynomials» [25]; 5.° «Linearizatsija proizvedenij klassicheskikh ortogonal'nykh polinomov» [26]; 6.° «New generating functions for a class of polynomials» [27], where the author, by continuing her recent work concerned with a new generalization by means of certain generating relations of several known polynomial systems belonging to the families of the classical Jacobi, Hermite and Laguerre polynomials, gives an interesting unification (and generalization) of the generating functions which include the mixed generating functions for Hermite and Laguerre polynomials, recently given by L. Carlitz; 7.° «Some polynomials of G. Panda and the polynomials related to them» [28]; 8.° «Asymptotik bei Jacobi-Polynomen und Jacobi-Funktionen» [29] «Some bilateral generating functions for a certain class of special functions» [30]; 9.° «Orthogonal functions on some permutation groups» [31]; and 10.° «The centralizer of the Laguerre polynomial set» [32], while in some of the authors' forthcoming papers various kinds of orthogonality for polynomials in several variables will be discussed by taking into consideration [14]-[16] and appropriate papers will deal with C. Brezinski's comprehensive results [17].

Classical results concerning the orthogonality of the Jacobi polynomials $\{(P_n^{(A,B)}(x)\}_{n=0}^{\infty}$ on $[-1, 1]$ with respect to $(1-x)^A(1+x)^B$ when $A, B > -1$, the Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ on $[0, \infty)$ with respect to $x^\alpha e^{-x}$ when $\alpha > -1$, and the Hermite polynomials $\{H_n^{(\mu)}(x)\}_{n=0}^{\infty}$ on $(-\infty, \infty)$ with respect to $|x|^{2\mu} e^{-x}$ when $\mu > -1/2$ have been well known for at least a century [2], [12]. With the appearance of the Bessel polynomials in 1949 [9], a problem arose concerning orthogonality: The Bessel polynomials weight function could not be found, even though its existence is guaranteed [1]. Instead, by using the Bessel polynomials moments, a complex weight function was exhibited, involving integration around the origin in the complex plane. The problem of finding a real weight function for the Bessel polynomials remains open to this day, although two other alternatives have been found recently [15], [10]. An important aspect of the paper [9], however, was that it rekindled interest in finding weight functions and in different kinds of orthogonality.

Until recently there were two kinds of orthogonality commonly accepted: that generated by a real weight function or real measure with suport on the real axis (the kind just alluded to at the beginning of the first paragraph), and that generated by a complex weight function with integration taken along a path in the complex plane. Indeed Geronimus [3] in 1931 showed that if a set of polynomials was orthogonal with respect to a real weight function $w(x)$ with compact support on the real axis, then it was also orthogonal in the complex sense with respect to the complex weight function

$$F(z) = \frac{1}{2\pi i} \int \frac{w(x)\, dx}{x - z}$$

so long as the path of integration encircled the support of $w(x)$. In particular the Jacobi polynomials are complex orthogonal even when $A, B \not> -1$ (see also [6]). The Bessel polynomials fit into this category even without the knowledge of a real weight function. Difficulty with the infinite support of the Laguerre and Hermite weight functions has slowed similar results for these sets, but some progress has been made [11], [13].

Geronmimus's paper leads to the additional question: How can the Jacobi, Laguerre, and Hermite polynomials be made orthogonal in the real sense when A, B, α, $2\mu \ngtr -1$? Boas' result guarantees they can, but actually finding a weight function has proved formidable.

By returning to the moments associated with these polynomials, which can easily be found even without a weight function [8], R. D. Morton and A. M. Krall [10] attacked the problem from the point of view of distribution theory. In so doing a third and new kind of orthogonality was found.

2. Distributional weight functions

Let w be a real weight function, or, more generally, a distribution which includes polynomials in its domain, and let $\langle w, \varphi\rangle$ denote the action of w on a test function φ. Let

$$\mu_n = < w , x^n > , \qquad n = 0, 1, \ldots ,$$

be the moments of w. If φ is analytic, then

$$< w , \rho > = < w , \sum_{n=0}^{\infty} \rho^{(n)} (0) x^n / n ! > = \sum_{n=0}^{\infty} < w\, x^n > \rho^{(n)} (0) / n ! =$$

$$= < \sum_{n=0}^{\infty} (- 1)^n \mu_n \delta^{(n)} / n' , \rho >$$

where $\delta^{(n)}$ is the nth distributional derivative of the Dirac delta function

$$w = \sum_{n=0}^{\infty} (- 1)^n \mu_n \delta^{(n)} / n' ,$$

in the sense of distributions. The classical representation of a weight function can be recovered from the series by using the inverse Fourier transform, summing the series, then using the Fourier transform [10]. We now show how the expansion above applies to the four classical polynomial sets mentioned in the introduction.

3. Jacobi polynomials

The Jacobi polynomials $\{P_n^{(A,B)}(x)\}_{n=0}^{\infty}$ satisfy the differential equation

$$(1 = x^2)\, y'' = B - A - (2 + A + Bx)\, y' + n\,(1 + A + B + n)\, y = 0$$

so long as A and B are not negative integers, where the polynomials degenerate. Through [8] and [10] the moments associated with these polynomials are

$$\mu_n = \sum_{j=0}^{u} \left(\frac{n}{j}\right) (-1)^j\, 2^j\, (A+1)_j / (A+B+2)_j =$$

$$= \sum_{j=0}^{n} \left(\frac{n}{j}\right) (-1)^{n-j}\, 2^j\, (B+1)_j / (A+B+2)_j.$$

Hence

$$w = \sum_{n=0}^{\infty} (-1)^u\, \mu_n\, \delta^{(u)} / n!$$

makes the polynomials orthogonal [10].

The inverse Fourier transform of w is

$$F^{-1} w = \frac{1}{2\pi} e^{it}\, {}_1F_1\,(B+1,\, A+B+2;\, -2it).$$

If A, B > -1, then

$$<w,\, \rho> = \frac{\Gamma(A+B+2}{\Gamma(A+1)\,\Gamma(B+1)\,2^{A+B+1}} \int_{-1}^{1} (1-1)^A (1+x)^B \rho(x)\, dx.$$

w is the standard weight function on $[-1, 1]$. If $-M-1 <$

$< B < - M$, $- N - 1 < A < - N$, M and N positive integers, then

$$< w, \rho > = \frac{\Gamma(A+B+2)}{\Gamma(A+1)\Gamma(B+1)2^{A+B+1}} \Bigg[\int_0^1 (1-x)^A \Bigg\{ (1+x)^B \rho(x) -$$

$$- \sum_{j=0}^{N-1} \frac{[(1+x)^B \rho(x)]^{(j)}}{j!} \Bigg|_{x=1} (-1)^j (1-x)^j \Bigg\} dx +$$

$$+ \int (1+x)^B \Bigg\{ (1-x)^A \rho(x) - \sum_{k=0}^{M-1} \frac{[(1-x)^A \rho(x)]^{(k)}}{k!} \Bigg|_{x=-1} (1+x^k) \Bigg\} dx +$$

$$+ \sum_{n=0}^{N-1} \frac{[(1+x)^B \rho(x)]^{(k)}}{k!} \Bigg|_{x-1} \frac{(-1)^j}{(A+j+1)} +$$

$$+ \sum_{k=0}^{M-1} \frac{[(1-x)^A \rho(x)]^{(k)}}{k!} \Bigg|_{x=-1} \frac{1}{(B+j+1)} \Bigg].$$

Here w is the canonical regularization of $(1-x)^A (1+x)^B$ (see [4]).

4. Laguerre polynomials

The Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$ satisfy the differential equation

$$x y'' - (x - \alpha - 1) y' + n y = 0$$

so long as α is not a negative integer, where the polynomials degenerate. Through [8] and [10], the moments associated with these polynomials are

$$\mu_n = \Gamma(n + \alpha + 1) / \Gamma(\alpha + 1).$$

Hence

$$w = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(n+\alpha+1) \delta^{(n)}}{\Gamma(\alpha+1) n!}.$$

Its inverse Fourier transform is

$$\mathrm{F}^{-1} w = \left(\frac{1}{2\pi}\right)(1+it)^{-\alpha-1}.$$

If $\alpha > -1$, then

$$< w, \rho > = \frac{\Gamma(\alpha+1)}{1} \int_0^\infty x^\alpha e^{-x} \rho(x)\, dx,$$

w is the standard weight function. If $-j-1 < \alpha < j$, j a positive integer, then

$$< w, \rho > = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty x^\alpha \Bigg[e^{-x} \rho(x) -$$

$$- \sum_{m=0}^{j-1} \left\{ \sum_{k=m}^{j-1} \frac{(-1)^k x^k}{m!\,(k-m)!} \right\} (-1)^m \rho^{(m)}(0) \Bigg] dx.$$

Here it is x which has been regularized [4].

5. Hermite polynomials

The Hermite polynomials $\{H_n^{(\mu)}(x)\}_{n=0}^{\infty}$ pose a more difficult problem since they satisfy two differential equations

$$x y'' + 2(\mu - x^2) y' + (2nx - \theta_n x^{-1}) y = 0,$$

where

$$\theta_{2n} = 0, \quad \theta_{2n+1} = 2\mu, \quad n = 0, 1, \ldots$$

Moments can be found, however, by observing that for $\mu > -1/2$,

$$\mu_n = \int_{-\infty}^{\infty} x^n |x|^{2\mu} e^{-x^2} dx.$$

These moments

$$\mu_{2n+1}=0,\quad \mu_{2n}=\Gamma\left(n+\mu+\frac{1}{2}\right),$$

generate $\{H_n^{(\mu)}(x)\}_{n=0}^{\infty}$ even when $\mu<-1/2$. Consequently

$$w=\sum_{n=0}^{\infty}\Gamma\left(n+\mu+\frac{1}{2}\right)\delta^{(2n)}/(2n)!$$

serves as a weight function so long as μ is not equal to

$$-\frac{1}{2},-\frac{3}{2},\ldots [7].$$

w has an inverse Fourier transform

$$F^{-1}w=\left(\frac{1}{2\pi}\right)\Gamma\left(\mu+\frac{1}{2}\right)_1F_1\left(\mu+\frac{1}{2},\frac{1}{2};-\frac{t^2}{4}\right).$$

When $\mu>-1/2$ it can be inverted to show

$$<w,\rho>=\int_{-\infty}^{\infty}|x|^{2\mu}e^{-x^2}\rho(x)\,dx,$$

the standard weight function. When

$$-2j-1<2\mu<-2j+1,$$

j a positive integer, the inversion shows [7]

$$<w,\rho>=\int_{0}^{\infty}|x|^{2\mu}\Big\{e^{-x^2}[\rho(x)+\rho(-x)]-$$

$$-\sum_{k=}^{2j-2}[e^{-x^2}(\rho(x)+\rho(-x))\quad \rho^{(k)}(0)x^k/k!\Big\}dx.$$

In this case there is a double regularization of $|x|^{2\mu}$ at $x=0$.

6. Bessel polynomials (4)

Bessel polynomials present a different and rather difficult problem. They satisfy the differential equation

$$x^2 y'' + (a x + b) y' - n (n + a - 1) y = 0 .$$

Consequently their moments [9] are

$$\mu_n = (-b)^{n+1} / (a)_n$$

which yield a distributional weight function

$$w = - \sum_{n=0}^{\infty} \frac{b^{n+1} \, \delta^{(n)}}{(a)_n \, n!} .$$

Its inverse Fourier transform, however, is

$$F^{-1} w = \frac{1}{2\pi} \sum_{n=}^{\infty} \frac{b^{n+1} (it)^n}{(a)_n \, n!} = \frac{-b \, \Gamma(a)}{2\pi (ibt)^{(a-1)/2}} I_{a-1} (\sqrt{4ibt}) .$$

To take the Fourier transform in even the distributional sense has so far proved impossible.

A complex weight function [3] has proved much easier to find. It is

$$F(z) = \sum_{n=0}^{\infty} \mu_n / z^{n+1} = - (b/z)_1 F_1 (1, a, - b/z) .$$

Using the Stieltjes inversion formula a distributional weight function can be computed [5], giving

$$< w, \rho > = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_a^b \operatorname{Im} \} (-b/z)_1 F_1 (1, a, -b/z) \, \rho(x) \{ dx ,$$

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(4) Another approaches and the two parameters generalized Bessel polynomials $y_n(z, a, b) = F(-n, n + a - 1; \; -n; -(z/b))$ are to be found among others in the Emil Grosswald's recent book: *Bessel polynomials*, Springer, New York, 182 p.

where

$$z = x + i\,\varepsilon, \quad \text{and} \quad \alpha < 0,\ \beta > 0,$$

but are otherwise arbitrary.

REMARKS.—1.° The idea of distributional weight functions can be applied to other polynomial sets. We hope that it will prove to be a useful tool in the future.

2.° In one of the authors' next papers our new look at some classic problems in orthogonal polynomials will take into consideration various B. A. Bondarenko, D. L. Fernández, D. Mangeron & al. researches [14], [15], [34] concerned with polyharmonic and polyvibrating or polywave orthogonal polynomials, while D. Mangeron and M. N. Oguztoreli extensions [16] of C. A. Truesdell «F-Functions» method, C. Brezinski's Padé — Type approximation and general orthogonal polynomials results [17] as well as D. Mangeron's studies concerning orthogonal Tchebycheff polynomials on multidimensional grids of equidistant points and «total derivatives operators» in the Picone's sense systems will be related with the authors «New Look» in a later research work.

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